HELSINKI LOGIC GROUP TALK 20 MAY 2020

TITLE & ABSTRACT

Measurable Selections:

A Bridge Between Large Cardinals and Scientific Applications?

Abstract: There is no prospect of discovering measurable cardinals by radio astronomy, say by locating a pulsar pumping out the digits of zero-sharp, but this does not mean that higher set theory is *entirely* irrelevant to and unconnected with applied mathematics broadly construed. By way of example, the bearing of some celebrated descriptive-set-theoretic consequences of large cardinals on measurable selection theory, a body of results originating with a key lemma von Neumann's work on the mathematical foundations of quantum theory, and further developed in connection with problems of mathematical economics, and one that perhaps deserves to be somewhat better known among logicians, will be considered from a philosophical point of view.

OUTLINE

Pragmatist stance sees truth-claims for mathematics as dependent on connections with applications, but with "applications" and "connection" understood broadly.

On breadth of "applications", philosophical points urged:
- theoretical as well as practical
- qualitative as well as quantitative
- auxiliary structures as well as main structures
- idealized as well as realistic models
  - value of model may lie in its very lack of realism

cases alluded to:
- Poincaré celestial mechanics (dynamical systems)
- Hamilton phase space
- Maxwell-Boltzmann ideal gas
- Einstein continuous model of diffusion in Brownian motion paper
- Carnot, Fourier caloric-fluid model in thermodynamics
- Schwarzschild, de Sitter, Gödel mathematical existence proofs
  - for models of general relativity
- Hardy-Weinberg law

On breadth of "connections", philosophical points urged:
- truth-claims extend downwards from applied results to axioms
  - axioms actually used, not truncated ones that could have been used
  - ("indispensability" not the criterion)
  - with classical mathematics, this means down to ZFC
- truth-claims also extend downwards from axioms to intuitive picture
  - with ZFC, this means down to iterative conception
- truth-claims extend upward from axioms even to unapplied consequences
- truth-claims extend upwards from intuitive picture to further axioms

examples of "intrinsic justification" of "small large cardinals" by iterative conception
- inaccessible in Zermelo (cf Grothendieck)
- Mahlo in Gödel (cf Friedman)
- weakly compact in Bernays
- but there are limits (Koellner)

On motivation of large large cardinals

- Maddy's rules of thumb:
  - *heuristics suggesting axioms worthy of exploration, not acceptance*
  - e.g. might be regarded as suggesting Reinhardt cardinal
- Cabalism's attractive consequences in descriptive set theory
  - classical result: measurability of Σ^1_1 and Π^1_1 (Souslin)
  - classical result: uniformization of Π^1_1 and Σ^1_2 (Kondô)
  - no more in ZFC (Addison, Levy)
  - measurable cardinals: measurability of Σ^1_2 and Π^1_2 (Solovay)
  - larger larger cardinals: projective measurability, uniformization (cabal)
  - *beauty is not truth, contra Keats*
- Gödel extrinsic justification through *verifiable* consequences
  - Martin's two proofs of Borel determinacy
    - measurable cardinal yields analytic, hence Borel determinacy
    - (actually Erdös cardinal, but measurable more natural axiom)
    - more difficult ZFC proof "verifies" Borel case
    - (cf. metatheoretic results of Friedman)
  - *and is there a connection with applications?*
- Measurable selection theorems, actual examples:
  - *Yankov / von Neumann* (structure of rings of operators)
    - analytic multifunctions have Lebesgue* measurable selections
    - *actually, universally
  - *Mauldin et al.* (mathematical economics / decision theory)
    - parameterized version of Debreu's representation theorem
    - (closed preference orders to continuous utility functions)
    - lemma derived from Harrington's proof of Silver's theorem
  - cf Wagner survey in SIAM journal for more
  - Kunisky shows work can be very much reduced using
    - measurable cardinal (or MA + not-CH)
    - Σ^1_2 multifunctions have measurable selections
    - work could be reduced still further if Σ^1_2 replaced by larger class
- *What is the most natural larger class and most natural large cardinal:*
    - *supercompact?*

TECHNICAL DEFITIONS

Let X,Y be a Polish (separable, complete-metrizable) spaces, e.g. reals
A *multifunction* from X to Y is a function $F: X \to \wp(Y)$
Graph(F) = $\{(x,y): y \in F(x)\}$
Multifunction is Borel, analytic, etc. iff Graph is
Domain(F) = $\{x: F(x) \neq \emptyset\}$,
i.e. projection of Graph(F)
A *selection* for F is a function f: Domain(F)$\to Y$ with $f(x) \in F(x)$,
i.e. graph(f) uniformizes Graph(F)
uniformization theory emphasizes definability properties,
selection theory emphasizes measurability properties
Yankov/von Neumann: every analytic multifunction has measurable selection
Kondô-Martin: assuming a measurable cardinal,
every PCA (alias Σ^1_2) multifunction has measurable selection

A *preference order* on X is a reflexive, transitive, connected relation $\leq$
it gives rise to a strict preference order and an equivalence relation
$x < y$ iff $x \leq y$ and not $y \leq x$ $\quad$ $x \equiv y$ iff $x \leq y$ and $y \leq x$
A *utility function* on X is a function $f: X \to [0,1]$
f represents $\leq$ if for all x,y we have $x \leq y$ iff $f(x) \leq f(y)$
Note that if f represents $\leq$ then so does the composition of f
with any monotone $g: [0,1] \to [0,1]$, so representation is never unique
$\leq$ is *closed* iff $\{y: y \leq x\}$ and $\{y: x \leq y\}$ are closed for all x
Debreu theorem:
Every closed preference order can be represented by a continuous utility function
Lemma needed for parameterized version:
If F is a coanalytic multifunction from X to X^2 such that each
$F(x)$ is an equivalence relation $\equiv_x$ and if $C = \{x: \equiv_x$ has countably many classes$\}$,
then there are measurable $f_i: C \to X$ for $i = 0,1,2,\ldots$
such that for all $x \in C$ and all y we have $y \equiv_x f_i(x)$ for some i
(Corollary of Harrington's proof of Silver's theorem)

REFERENCES

[ITEMS ON MEASURABLE SELECTIONS ONLY]

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