FINITARY ABSTRACT ELEMENTARY CLASSES

Meeri Kesälä

Academic dissertation

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Helsinki, November 2006

Meeri Kesälä
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INTRODUCTION

In this doctoral thesis we introduce finitary abstract elementary classes, a non-elementary framework of model theory. These classes are a special case of abstract elementary classes (AEC), introduced by Saharon Shelah [26] in the 1980’s. We have collected a set of properties for classes of structures, which enables us to develop a ‘geometric’ approach to stability theory, including an independence calculus, in a very general framework. The novelty is the property of finite character, which enables to use weak type as a notion of type. The thesis consists of three independent papers. All three papers are joint work with Tapani Hyttinen.

I Independence in finitary abstract elementary classes,
Tapani Hyttinen and Meeri Kesälä.

II Categoricity transfer in simple finitary abstract elementary classes,
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Paper I will appear in the journal Annals of Pure and Applied Logic. The first versions of these papers were written at the following times: Paper I during the years 2004 and 2005, Paper II in autumn 2005 and Paper III in spring 2006. Part of the material in Paper I is presented in the author’s licentiate thesis Independence in Local Abstract Elementary Classes, 2005. All the work for this thesis was done at the University of Helsinki and the author was supported by the graduate school MALJA.

During the last decades of the twentieth century, the focus in model theory moved from the study of syntactical questions to the study of structural properties of classes of models of a theory. An elementary class Mod(T) is the collection of models of similarity type τ satisfying the axioms of a complete theory T in elementary logic.
with vocabulary \( \tau \). The work by Morley and Shelah played an essential role in this transformation.

The Löwenheim-Skolem theorem of elementary logic says that a theory with infinite models has models in every infinite cardinality above the size of the vocabulary. Hence we cannot hope that the theory could describe an infinite model up to isomorphism. But can the theory describe a model uniquely in a fixed cardinality? We say that a class \( \text{Mod}(T) \) is \( \lambda \)-categorical, if it has only one model of cardinality \( \lambda \), up to isomorphism. A famous theorem of by Michael D. Morley [21] says that if the class of models of theory in a countable elementary language is \( \lambda \)-categorical in some uncountable cardinal \( \lambda \), it is categorical in every uncountable cardinal. The proof of this theorem introduced useful methods for classifying structures such as ranks and counting the number of types in a structure. Shelah developed the theory further by introducing a wide collection of tools, such as a general notion of independence and a concept of a strong type, see the book [23]. On the basis of the number of types of tuples over a set of a fixed size, we can divide theories into different classes, so called \( \aleph_0 \)-stable, superstable, stable or unstable theories. Shelah’s Main Gap Theorem introduces a dividing line for classes of models of a countable and complete theory: it says that there are either the maximal number of models very hard to distinguish from each other, or a relatively small number of relatively easily distinguishable models in each cardinality \( \aleph_\alpha \). The proof of this theorem uses stability theory and properties of the independence calculus.

Many natural classes of structures in mathematics are not axiomatizable by means of elementary logic. In order to generalize classification theory to a wider range of classes, many non-elementary frameworks have been introduced. An important one is classes of structures definable in the language \( L_{\omega_1\omega} \), where countable conjunctions and disjunctions are allowed. Shelah defined excellent classes in [25]. There he studies a class of models of a \( L_{\omega_1\omega} \)-sentence, which has good amalgamation properties. Another extensively studied context is homogeneous classes [22] where one studies elementary substructures of a big homogeneous model. A characteristic for the non-elementary contexts is the failure of compactness, which makes the classification task more difficult. However, both in the excellent and the homogeneous framework there are good analogues to Morley’s theorem. Also analogues to the Main Gap have been studied; see [7] for excellent classes by Grossberg and Hart and [17] for a homogeneous framework by Hyttinen and Shelah. Both contexts of homogeneous classes and of excellent classes have been used for applications in concrete classes; see for example Berenstein and Buechler [5], Mekler and Shelah [20] or Zilber [30], [29].
The independence calculus

Baldwin lists in the book [2] the essential properties of a notion of independence ↓. We call such properties an independence calculus. These properties hold for the notion based on nonforking in stable first-order theories. If the theory $T$ is superstable or $\aleph_0$-stable, we have in addition that $\kappa(T) = \aleph_0$ in local character below. We write $\bar{a} \downarrow_A B$ and say that the type $\text{tp}(\bar{a}/B \cup A)$ is independent over $A$.

1. **Invariance:** If $f$ is an isomorphism and $\bar{a} \downarrow_A B$, then $f(\bar{a}) \downarrow_{f(A)} f(B)$.
2. **Monotonicity:** If $A \subset B \subset C \subset D$ and $\bar{a} \downarrow^*_A D$, then $\bar{a} \downarrow^*_B C$.
3. **Transitivity:** Let $B \subset C \subset D$. If $\bar{a} \downarrow_B C$ and $\bar{a} \downarrow_C D$, then $\bar{a} \downarrow_B D$.
4. **Symmetry:** If $\bar{a} \downarrow_A \bar{b}$, then $\bar{b} \downarrow_A \bar{a}$.
5. **Extension:** For any tuple $\bar{a}$ and $A \subset B$ there is a type $\text{tp}(\bar{b}/B)$ extending $\text{tp}(\bar{a}/A)$ such that $\bar{b} \downarrow_A B$.
6. **Finite character:** If $\bar{a} \notin_A B$ and $A \subset B$, there is a formula $\phi(x, \bar{b}) \in \text{tp}(\bar{a}/B)$ such that no type containing $\phi(x, \bar{b})$ is independent over $A$.
7. **Local character:** There is a cardinal $\kappa(T)$ such that for any $\bar{a}$ and $B$ there is $A \subset B$ such that $|A| < \kappa(T)$ and $\bar{a} \downarrow_A B$.
8. **Reflexivity:** If $A \subset B$, $\bar{b} \in B \setminus A$ and $\text{tp}(\bar{b}/A)$ is not algebraic, then $\bar{b} \notin_A B$.
9. **Stationarity:** Assume that $\mathcal{A}$ is a model, $\text{tp}(\bar{a}/\mathcal{A}) = \text{tp}(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow^*_B \mathcal{A}$ and $\bar{b} \downarrow^*_B \mathcal{A}$. Then $\text{tp}(\bar{a}/B) = \text{tp}(\bar{b}/B)$.

It might be useful to have at least some of the above properties for independence or some restricted forms of the properties. See for example Shelah [24] for the study of simple and unstable first-order theories.

Abstract elementary classes

Shelah suggested in [26] abstract elementary classes (AEC) as a platform to study model theoretic concepts in a more general setting. He studies a class $\mathbb{K}$ of structures in a fixed similarity type $\tau$, but does not define any specific language. Instead he gives axioms for $(\mathbb{K}, \preceq_{\mathbb{K}})$, where $\preceq_{\mathbb{K}}$ is a relation between the models of the class. The class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is for example assumed to be closed under isomorphisms, behave well with respect to $\preceq_{\mathbb{K}}$-increasing chains and have a downward Löwenheim-Skolem number $\text{LS}(\mathbb{K})$. If $B$ is a subset of a structure $\mathcal{A} \in \mathbb{K}$, there is a $\preceq_{\mathbb{K}}$-elementary substructure of $\mathcal{A}$ containing $B$ of cardinality at most $\text{LS}(\mathbb{K}) + |B|$. This context generalizes an elementary class $(\text{Mod}(T), \preceq)$, where $\preceq$ is the elementary substructure relation.

In his Presentation Theorem Shelah showed that such a class can be presented as a class of reducts of models in an elementary class omitting a set of types. This
enables us to use the method of Ehrenfeucht-Mostowski models in the study of AEC’s and gives a Hanf number depending on the Löwenheim-Skolem number of the class. If an abstract elementary class has models of cardinality greater or equal to the Hanf number, it has arbitrarily large models. Shelah also stated a conjecture: There is a cardinal $\kappa$ such that if an abstract elementary class is categorical in one cardinal above $\kappa$, then it is categorical in all cardinals above $\kappa$.

**Galois types**

If we want to generalize more model theoretic tools to abstract elementary classes, we might have to isolate the needed properties from elementary model theory as new axioms for the class. For example, there is a problem of defining a good notion of type. The most popular procedure is to assume the amalgamation property and use a notion of Galois type. Galois types generalize the usual notion of types, defined as sets of formulas. The concept is due to Shelah, but the name Galois type was introduced by Grossberg [6]. Another common practice is to assume the class to have arbitrarily large models, amalgamation and joint embedding properties. Then we can use the construction by Jónsson and Fraïssé [18] to build a universal and model-homogeneous *monster model* $\mathcal{M} \in \mathbb{K}$. Two tuples in $\mathcal{M}$ have the same Galois type over a model $\mathcal{A}$, if there is an automorphism of the monster model mapping $\bar{a}$ to $\bar{b}$ and fixing $\mathcal{A}$ pointwise.

In [27], Shelah shows that in the framework above there is a cardinal $H_2$ called the second Hanf number\(^1\), such that if $H_2 < \lambda \leq \kappa$ and the class is categorical in the successor cardinal $\kappa^+$, then it is categorical in $\lambda$. Shelah showed that categoricity above $H_2$ implies some good behaviour for Galois types, which enables the transfer of categoricity. Grossberg and VanDieren [10] [11] [12] isolated the notion of *tameness* of Galois types as the required property for categoricity transfer. Let $\text{tp}_g(\bar{a}/\mathcal{A})$ denote the Galois type of a tuple $\bar{a}$ in the monster model over an $\preceq_\mathbb{K}$-elementary submodel $\mathcal{A}$. A class is said to be tame in a cardinal $\chi$, if for all models $\mathcal{A}$ such that $\text{tp}_g(\bar{a}/\mathcal{A}) \neq \text{tp}_g(\bar{b}/\mathcal{A})$, there is $\mathcal{B} \preceq_\mathbb{K} \mathcal{A}$ of size at most $\chi$ such that $\text{tp}_g(\bar{a}/\mathcal{B}) \neq \text{tp}_g(\bar{b}/\mathcal{B})$.

We say that a class is tame, if it is tame in $\text{LS}(\mathbb{K})$. The context of [10] is an abstract elementary class with amalgamation, joint embedding, arbitrary large models and tameness in some cardinal $\chi$. Then categoricity in some $\kappa^+ > \max\{\chi, \text{LS}(\mathbb{K})^+\}$ implies categoricity in all $\lambda \geq \kappa^+$. Lessmann [19] showed that categoricity can be

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\(^1\)Baldwin [1] improves this result by replacing $H_2$ with the Hanf number.
transferred upwards also from $\aleph_1 > \max\{\chi, \text{LS}(\mathbb{K})\}$. Both these results and the result of Shelah’s require the categoricity cardinal to be a successor.

**Independence in abstract elementary classes**

Several authors besides Shelah, including Baldwin, Grossberg, Kolesnikov, Lessmann, VanDieren and Villaveces have studied abstract elementary classes. John Baldwin has collected much of the current research in his book [1]. Also in the paper [3] he has listed several open questions of the field. Among these questions is to study an independence calculus and find the ‘correct’ notion of superstability for AEC’s. Some notions of independence have been introduced for AEC’s, see Shelah [27] or Grossberg [6], but the analogue of the full independence calculus in elementary classes has not been achieved in the most general context. Many examples of AEC’s, including excellent classes and homogeneous classes, admit a notion of independence. There are also several frameworks of AEC’s with a abstract notion of independence, where the definition is not specified but only axioms for the independence calculus are given; see for example Shelah [28], Grossberg and Kolesnikov [8] or Grossberg and Lessmann [9].

In saying that a tuple $\bar{a}$ is independent of a set $B$ over a set $C$, written $\bar{a} \downarrow_C B$, we mean roughly that the set $B$ does not give more information about $\bar{a}$ than $C$ does. In the following examples of AEC’s the ‘natural’ notion of independence agrees with a model-theoretic notion in a suitable framework. Example 1 is elementary and for Example 2 in an homogeneous context see Berenstein and Buechler [5].

**Example 1** (Field of complex numbers $\mathbb{C}$). Consider the class of algebraically closed fields of characteristic zero and take as the notion $\preceq_K$ the subfield relation. Denote by $\text{acl}(A)$ the algebraic closure of a subset $A$ of an algebraically closed field. Then for any subsets $C \subset B$, $\bar{a} \downarrow_C B$ iff

$$\text{acl}(\bar{a} \cup C) \cap \text{acl}(B) \subset \text{acl}(C).$$

**Example 2** (Hilbert spaces). Consider the class of all normed vector spaces over the reals which can be completed to a Hilbert space. We take as the notion $\preceq_K$ the linear subspace relation. Denote by $\bar{B}$ the closed subspace generated by $B$ and by $P_B(\bar{a})$ the orthogonal projection of the tuple $\bar{a}$ to the space $\bar{B}$. Let $A \subset B$ be subsets of a Hilbert space. Then $\bar{a} \downarrow_A B$ iff

$$P_B(\bar{a}) \in \bar{A}.$$

That is, when we write $\bar{a} = \bar{a}_A + \bar{a}_\perp$, where $\bar{a}_A \in \bar{A}$ and $\bar{a}_\perp$ is in the orthocomplement of $\bar{A}$, then $\bar{a} \downarrow_A B$ if and only if $\bar{a}_\perp$ is orthogonal to $\bar{B}$. 
In this thesis we introduce a context of finitary classes, with the relation \( \preceq_k \) having finite character. Let \( \text{tp}^g(\bar{a}/\emptyset, \mathcal{A}) \) denote the Galois type of a tuple \( \bar{a} \) in some model \( \mathcal{A} \in \mathcal{K} \) over the empty set. We define finite character as the property that if \( \mathcal{A}, \mathcal{B} \in \mathcal{K} \), \( \mathcal{A} \subseteq \mathcal{B} \) and for each finite tuple \( \bar{a} \in \mathcal{A} \) we have that
\[
\text{tp}^g(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^g(\bar{b}/\emptyset, \mathcal{B}),
\]
then \( \mathcal{A} \preceq_k \mathcal{B} \). Finite character is not a form of compactness, since it describes the relation between two models in the class, and is not a way to construct new models. If the relation \( \preceq_k \) is defined syntactically in a logic \( L \) which has only finitely many free variables in a formula, then it has finite character. Both the excellent and homogeneous classes belong to this framework.

We say that a notion of type has finite character when any two tuples have the same type over a set \( A \) if and only if they have the same type over each finite subset of \( A \). In the elementary context the notion of Galois type agrees with that of syntactic type, and thus has finite character. This is essential in many constructions of elementary classification theory and enables us to define a notion of independence based on dependency of finite sets. We want to study particularly this kind of finite dependencies and take as the notion of type that of weak type, which has finite character by definition.

The two main themes in all three papers in the thesis are ‘How good a control can we have on the behaviour of Galois types?’ and ‘Can we build a good analogue for the independence calculus in elementary classes?’ For the first question the answer remains unsatisfactory, since we often have to assume tameness for Galois types. The study of the second question is more rewarding.

I Independence in finitary abstract elementary classes

In the first paper we define finitary classes to be abstract elementary classes \((\mathcal{K}, \preceq_k)\) with countable Löwenheim-Skolem number, arbitrarily large models, disjoint amalgamation, prime model and finite character. Disjoint amalgamation and prime model are slightly stronger versions of amalgamation and joint embedding. By the Jónsson-Fraïssé construction we can build a monster model. This model has an expansion in the elementary class implied by Shelah’s Presentation theorem. Using the assumptions of disjoint amalgamation and prime model we can make the expansion to be a homogeneous model. The stronger versions of amalgamation are only needed for this, and we use the homogeneous expansion to gain good control over indiscernible sequences. We define weak type of finite tuples as follows:
\[
\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{b}/A) \text{ iff } \text{tp}^g(\bar{a}/B) = \text{tp}^g(\bar{b}/B) \text{ for all finite } B \subset A.
\]
In the two first papers we restrict the study on the \( \aleph_0 \)-stable case. We define \( \aleph_0 \)-stability with respect to weak types, but are able to show that this notion is equivalent to the notion for Galois types. We prove as Theorem 3.12 the following:

**Theorem 3.** Assume that \((\mathcal{K}, \preceq_\mathcal{K})\) is finitary and \( \aleph_0 \)-stable with respect to weak types. Let \( \mathcal{A} \) be a countable model, \( \bar{a} \) and \( \bar{b} \) finite tuples and \( \text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \). Then also \( \text{tp}^0(\bar{a}/\mathcal{A}) = \text{tp}^0(\bar{b}/\mathcal{A}) \).

The proof for the theorem is a primary model construction. It follows that under \( \aleph_0 \)-stability and \( \aleph_0 \)-tameness the notions of weak type and Galois type agree over all models of \( \mathcal{K} \).

We define two different notions of independence. The first is denoted by \( \lup^s \) and the definition is based on splitting over finite sets. We are able to show several properties for \( \lup^s \) over \( \aleph_0 \)-saturated models using the finite character of \((\mathcal{K}, \preceq_\mathcal{K})\) and \( \aleph_0 \)-stability. To gain the full picture including symmetry, we need to assume extension property for splitting. This assumption is needed throughout the paper, and we show that it is implied either from tameness or categoricity above the Hanf number. The properties are listed in Theorem 3.17 and Corollaries 4.14 and 4.21 of Paper I. Reflexivity is an easy consequence of the definition and it is not mentioned in the paper, but we list it here for completeness.

**Theorem 4.** Assume that \((\mathcal{K}, \preceq_\mathcal{K})\) is an \( \aleph_0 \)-stable finitary AEC. Then \((\mathcal{K}, \preceq_\mathcal{K})\) has a notion of splitting with the following properties:

1. **Invariance:** If \( f \) is an automorphism of the monster model \( \mathfrak{M} \), \( \bar{a} \lup^s_A B \) if and only if \( f(\bar{a}) \lup^s_{f(A)} f(B) \).
2. **Monotonicity:** If \( A \subset B \subset C \subset D \) and \( \bar{a} \lup^s_A D \), then \( \bar{a} \lup^s_B C \).
3. **Transitivity:** If \( A \subset B \subset C \) and \( \mathcal{B} \) is an \( \aleph_0 \)-saturated model, then \( \bar{a} \lup^s_A C \) if and only if \( \bar{a} \lup^s_A B \) and \( \bar{a} \lup^s_B C \).
4. **Countable extension:** Let \( \mathcal{A} \subset B \) be countable and let \( \mathcal{A} \) \( \aleph_0 \)-saturated model. For each \( \bar{a} \) there is \( \bar{b} \) realizing \( \text{tp}^w(\bar{a}/\mathcal{A}) \) such that \( \bar{b} \lup^s_{\mathcal{A}} B \). Moreover, if \( \text{tp}^w(\bar{a}/\mathcal{A}) \) does not split over some finite subset \( E \subset \mathcal{A} \), then \( \text{tp}^w(\bar{b}/B) \) does not split over \( E \).
5. **Finite character:** Let \( \mathcal{A} \) be an \( \aleph_0 \)-saturated model and \( \mathcal{A} \subset B \). Then \( \bar{a} \lup^s_{\mathcal{A}} B \) if and only if \( \bar{a} \lup^s_{\mathcal{A}} B_0 \) for every finite \( B_0 \subset B \).
6. **Local character:** For each model \( \mathcal{A} \) and finite sequence \( \bar{a} \) there is a finite \( E \subset \mathcal{A} \) such that \( \bar{a} \lup^s_{\mathcal{A}} \).
7. **Reflexivity:** Let \( \mathcal{A} \) be an \( \aleph_0 \)-saturated model. Then for all tuples \( \bar{a} \notin \mathcal{A} \), \( \bar{a} \lup^s_{\mathcal{A}} \).
8. **Stationarity:** Assume \( \mathcal{A} \) is an \( \aleph_0 \)-saturated model and \( A \subset B \). If \( \text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \), \( \bar{a} \lup^s_{\mathcal{A}} B \) and \( \bar{b} \lup^s_{\mathcal{A}} B \), then \( \text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B) \).
Furthermore, if \((\mathbb{K}, \preceq_{\mathbb{K}})\) is in addition tame or categorical above the Hanf number:

8. **Extension:** Let \(\mathcal{A}\) be an \(\aleph_0\)-saturated model and \(\mathcal{A} \subset B\). For each \(\bar{a}\) there is \(\bar{b}\) realizing \(tp^w(\bar{a}/\mathcal{A})\) such that \(\bar{b} \downarrow_{\mathcal{A}} B\). Moreover, if \(tp^w(\bar{a}/\mathcal{A})\) does not split over some finite subset \(E\), then \(tp^w(\bar{b}/B)\) does not split over \(E\).

9. **Symmetry:** Let \(\mathcal{A}\) be an \(\aleph_0\)-saturated model. Then \(\bar{a} \downarrow_{\mathcal{A}} \bar{b}\) if and only if \(\bar{b} \downarrow_{\mathcal{A}} \bar{a}\).

We define weak \(\lambda\)-stability and weak saturation with respect to weak types. We show that \(\aleph_0\)-stability implies weak stability in each cardinality in finitary classes. If we assume also the extension property 4(8), we are able to show that weakly saturated models exist in every cardinality, by showing that the union of weakly saturated models is weakly saturated. We gain the analogous results for Galois types assuming tameness.

The second notion of independence, denoted with the symbol \(\downarrow\), is based on Lascar splitting. Lascar splitting is a version of strong splitting in elementary classes, which generalizes to the non-elementary context. We also define Lascar strong type of a tuple \(\bar{a}\) in the monster model over a set \(A\), written \(\Lstp(\bar{a}/A)\), such that for any \(n\)-tuples \(\bar{a}\) and \(\bar{b}\),

\[\Lstp(\bar{a}/A) = \Lstp(\bar{b}/A)\]

if \((\bar{a}, \bar{b}) \in E\) for each \(A\)-invariant equivalence relation \(E\) of \(n\)-tuples with a bounded number of classes. For the notion \(\downarrow\) we get all the usual properties of the independence calculus over sets, assuming \(\aleph_0\)-stability, the extension property and simplicity. The study on Lascar splitting and simplicity is an analogue to the similar study in excellent classes, see Hyttinen and Lessmann [15]. Simplicity is defined as the notion \(\downarrow\) having local character for arbitrary sets (see below), but it is enough to assume local character for finite sets only, as is done in Paper III. Without simplicity there might not be any notion of independence with these properties over sets. Shelah has provided such an example, see Hyttinen and Lessmann [16]. The following properties are listed in Paper I as Theorem 6.5, except reflexivity, which follows from Lemma 5.38(b).

**Theorem 5.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is finitary, simple, stable in \(\aleph_0\) and has the extension property. Then, \(\downarrow\) satisfies the following properties:

1. **Invariance:** If \(A \downarrow C B\), then \(f(A) \downarrow_{f(C)} f(B)\) for any automorphism \(f\) of the monster model.
2. **Monotonicity:** If \(A \downarrow B D\) and \(B \subset C \subset D\) then \(A \downarrow C D\) and \(A \downarrow B C\).
3. **Transitivity:** Let \(B \subset C \subset D\). If \(A \downarrow B C\) and \(A \downarrow C D\), then \(A \downarrow B D\).
4. **Symmetry:** \(A \downarrow C B\) if and only if \(B \downarrow C A\).
(5) **Extension:** For any $\bar{a}$ and $C \subset B$ there is $\bar{b}$ such that $\text{tp}^w(\bar{b}/C) = \text{tp}^w(\bar{a}/C)$ and $\bar{b} \downarrow_C B$.

(6) For any finite $C$, $\bar{a}$ and $B$ containing $C$, there is $\bar{b}$ such that $\text{Lstp}(\bar{b}/C) = \text{Lstp}(\bar{a}/C)$ and $\bar{b} \downarrow_C B$.

(7) **Finite character:** $A \downarrow_C B$ if and only if $\bar{a} \downarrow_C \bar{b}$ for every finite $\bar{a} \in A$ and $\bar{b} \in B$.

(8) **Local character:** For any finite $\bar{a}$ and any $B$ there exists a finite $E \subset B$ such that $\bar{a} \downarrow_E B$.

(9) **Reflexivity:** For each $\bar{a}$ and $C$ such that $\text{tp}^w(\bar{a}/C)$ is not bounded, $\bar{a} \notin_C \bar{a}$.

(10) **Stationarity over $\aleph_0$-saturated models:** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. If $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow_{\mathcal{A}} B$ and $\bar{b} \downarrow_{\mathcal{A}} B$, then $\text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B)$.

(11) **Stationarity of Lascar strong types:** If $\text{Lstp}(\bar{a}/C) = \text{Lstp}(\bar{b}/C)$, $\bar{a} \downarrow_C B$ and $\bar{b} \downarrow_C B$, then $\text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B)$.

We also define $U$-rank in $\aleph_0$-stable finitary classes with the extension property, and show that finite $U$-rank implies simplicity. First we define inductively the $U$-rank of a tuple $\bar{a}$ over an $\aleph_0$-saturated countable model $\mathcal{A}$, written $U(\bar{a}/\mathcal{A})$. Always $U(\bar{a}/\mathcal{A}) \geq 0$ and $U(\bar{a}/\mathcal{A}) \geq \alpha + 1$ if there is a countable $\aleph_0$-saturated $\mathcal{B} \supseteq \mathcal{A}$ such that $U(\bar{a}/\mathcal{B}) \geq \alpha$ and $\bar{a} \notin^*_{\mathcal{B}} \mathcal{B}$. The $U$-rank over an arbitrary $\aleph_0$-saturated model is defined as a minimum of $U$-ranks over countable $\aleph_0$-saturated submodels. Finally we generalize the definition to ranks over arbitrary sets as is done by Hyttinen and Lessmann in [15].

Finite character is essential both for the independence calculus in Theorems 4 and 5 and the proof of Theorem 3. The framework of this paper is further studied by Hyttinen in [14].

### II Categoricity transfer in simple finitary abstract elementary classes

In the second paper we introduce a weaker set of axioms for finitary classes, and show that all the main results of Paper I hold also with the weaker assumptions. We assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an abstract elementary class with a countable Löwenheim-Skolem number, arbitrarily large models, finite character, amalgamation and joint embedding. The use of the homogeneous expansion of the monster model is replaced by a finer study on Ehrenfeucht-Mostowski models. We also refine the study of Paper I on $U$-rank and equivalents of the extension property.

We define a notion of a constructible model called an f-primary model. Such models exist over any set by simplicity. We also need to assume $\aleph_0$-stability and
the extension property for splitting, but we show that simplicity and weak categoricity in any uncountable cardinal imply both of these. We say that the class $\mathbb{K}$ is weakly $\lambda$-categorical if all models of size $\lambda$ are weakly saturated. We use $f$-primary models to prove the following in Theorem 4.11. Here $(\mathbb{K})^\omega$ is the class of $\aleph_0$-saturated models of $\mathbb{K}$.

**Theorem 6.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary AEC and weakly categorical in some uncountable cardinal $\kappa$. Then

1. $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is weakly categorical in each uncountable $\kappa$ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in each $\lambda$ such that $\lambda \geq \min\{\kappa, \text{Hanf}\}$.

We denote by Galois saturation the saturation respect to Galois types. Under tameness the notions of weakly saturated and Galois saturated agree, and we have that any two Galois saturated models of equal cardinality are isomorphic. Thus the previous theorem gives a categoricity transfer result for $\aleph_0$-tame simple finitary classes, with no restrictions on the cofinality of the categoricity cardinal.

**Corollary 7.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple tame finitary AEC categorical in some uncountable $\kappa$. Then

1. $((\mathbb{K})^\omega, \preceq_{\mathbb{K}})$ is categorical in each uncountable $\kappa$ and
2. $(\mathbb{K}, \preceq_{\mathbb{K}})$ is categorical in each $\lambda$ such that $\lambda \geq \min\{\kappa, \text{Hanf}\}$.

An example introduced by Hart and Shelah [13] and further studied by Baldwin and Kolesnikov [4], shows that tameness is necessary for the categoricity transfer. The example shows that for each finite $k > 0$ there is a finitary class which is categorical in the cardinals $\aleph_0, ..., \aleph_k$, but fails categoricity in $\aleph_{k+1}$.

III Superstability in simple finitary AEC

In Paper III the definition of a finitary class is as in Paper II. We introduce a notion of superstability, which denies the existence of infinite ‘forking chains’. If $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an elementary class, the notion coincides with the usual notion. We show that we gain the independence calculus for $\downarrow$ over finite sets and arbitrary models, assuming both simplicity and superstability. If we add another assumption called the Tarski-Vaught property, we gain the independence calculus over all sets. This result improves also the result of Paper II, since simple and $\aleph_0$-stable finitary classes are superstable and have the Tarski-Vaught property. We do not need to assume the extension property for splitting. The full independence calculus is stated in Theorem 3.13 of Paper III.

**Theorem 8.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple, superstable, finitary AEC with the Tarski-Vaught property. Then the relation $\downarrow$ has the following properties.
(1) **Invariance:** If \( A \downarrow_C B \) and \( f \) is an automorphism of the monster model, then \( f(A) \downarrow_{f(C)} f(B) \).

(2) **Monotonicity:** If \( A \downarrow_B D \) and \( B \subset C \subset D \) then \( A \downarrow_C D \) and \( A \downarrow_B C \).

(3) **Transitivity:** Let \( B \subset C \subset D \). If \( A \downarrow_B C \) and \( A \downarrow_C D \), then \( A \downarrow_B D \).

(4) **Symmetry:** \( A \downarrow_C B \) if and only if \( B \downarrow_A A \).

(5) **Extension:** For any \( \bar{a} \) and \( C \subset B \) there is \( \bar{b} \) such that \( Lstp^w(\bar{b}/C) = Lstp^w(\bar{a}/C) \) and \( \bar{b} \downarrow_C B \).

(6) **Finite character:** \( A \downarrow_C B \) if and only if \( \bar{a} \downarrow_C \bar{b} \) for every finite \( \bar{a} \in A \) and \( \bar{b} \in B \).

(7) **Local character:** For any finite \( \bar{a} \) and any \( B \) there exists a finite \( E \subset B \) such that \( \bar{a} \downarrow_E B \).

(8) **Reflexivity:** If \( tp^w(\bar{a}/A) \) is not bounded, then \( \bar{a} \not\in_A \bar{a} \).

(9) **Stationarity:** If \( Lstp^w(\bar{a}/C) = Lstp^w(\bar{b}/C) \), \( \bar{a} \downarrow_C B \) and \( \bar{b} \downarrow_C B \), then \( Lstp^w(\bar{a}/B) = Lstp^w(\bar{b}/B) \).

Two tuples \( \bar{a} \) and \( \bar{b} \) have the same weak Lascar strong type over a set \( A \), written

\[
Lstp^w(\bar{a}/A) = Lstp^w(\bar{b}/A),
\]

if \( Lstp(\bar{a}/B) = Lstp(\bar{b}/B) \) for each finite \( B \subset A \). We show that with the assumptions above, the equivalence of weak Lascar strong types implies the equivalence of Galois types over any countable set. Again \( \aleph_0 \)-tameness generalizes this result to types over arbitrary models, and we are able to determine the Galois type by finitary means. Furthermore, we are able to apply stationarity of Theorem 8(9) not only to gain equivalence of weak Lascar strong types but also to gain equivalence of Galois types.

We define that a model \( \mathcal{A} \) is \textit{\( \alpha \)-saturated} if all Lascar strong types over finite subsets are realized in \( \mathcal{A} \). The following is Theorem 3.21 of Paper III.

**Theorem 9.** Assume that \((\mathcal{K}, \preceq)\) is tame, simple, superstable, finitary AEC with the Tarski-Vaught property. If \( \mathcal{A} \) is an \( \alpha \)-saturated model, then the following are equivalent:

1. \( Lstp^w(\bar{a}/\mathcal{A}) = Lstp^w(\bar{b}/\mathcal{A}) \)
2. \( tp^g(\bar{a}/\mathcal{A}) = tp^g(\bar{b}/\mathcal{A}) \)
3. \( tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A}) \).

We define a concept of \textit{\( \alpha \)-categoricity} in a cardinal \( \kappa \) as the property that there is only one \( \alpha \)-saturated model of size \( \kappa \), up to isomorphism. We show that superstability is implied by \( \alpha \)-categoricity in a cardinal \( \kappa \) above the Hanf number with \( cf(\kappa) > \omega \). As an application, we prove an \( \alpha \)-categoricity transfer result using \( \alpha \)-primary models.
Theorem 10. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple, tame finitary AEC with the Tarski-Vaught property. If $(\mathbb{K}, \preceq_{\mathbb{K}})$ is $\kappa$-categorical in some $\kappa \geq \text{Hanf}$ with uncountable cofinality, it is $\kappa$-categorical in any $\kappa \geq \text{Hanf}$.

As part of the proof we show that under simplicity and superstability, in all large enough cardinalities there is a model $\mathcal{A}$ such that all weak Lascar strong types over subsets of size $< |\mathcal{A}|$ are realized in $\mathcal{A}$. Again this is done by showing that an arbitrary union of such models has the same property.

The notion of superstability is tailored for simple, finitary classes and the results rely heavily on these properties. Several proofs use trees or other constructions of finite sets and they cannot be applied in a context without finite character. The question about a notion of superstability for general AEC remains open. However, this framework can be thought as a generalization of the context of excellent classes beyond $\aleph_0$-stability.

In conclusion, it seems that finitary classes provide a good platform for generalizing the theory of independence to a non-elementary context, and give many reasons for further study. We have studied the superstable case, but one could try to study the theory assuming only weak stability, and maybe simplicity. One might try to formulate a stability hierarchy theorem for weak types. Also one could try to find a classification for finitary classes with some analogue of the Main Gap theorem.

One other direction is to analyze further the $\aleph_0$-stable case and some context ‘near excellence’. The notion of a primary model is important in excellent classes. We introduce several notions of primary models for finitary classes, but we are not able to show similar good properties for these notions. Assuming $\aleph_0$-stability and $\aleph_0$-tameness we could try to prove for example uniqueness for some notion of primary model. Also there are some contexts where $\aleph_0$-tameness has been proved from the existence of a well-behaved notion of independence and a notion of amalgamation over independent $\mathcal{P}^-(n)$-diagrams, see Grossberg and Kolesnikov [8]. Could we find ‘a notion of excellence’ for finitary classes?

References

[13] Bradd Hart and Saharon Shelah. Categoricity over $P$ for first order $T$ or categoricity for $\phi \in L_{\omega,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$. *Israel J Math*, 70:219–235, 1990. Shelah [HaSh:323].


PAPER I

INDEPENDENCE IN FINITARY ABSTRACT ELEMENTARY CLASSES

_Tapani Hyttinen and Meeri Kesälä_

INDEPENDENCE IN FINITARY ABSTRACT ELEMENTARY
CLASSES

TAPANI HYTTINEN AND MEERI KESÄLÄ

Abstract. In this paper we study specific subclasses of abstract elementary
classes. We construct a notion of independence for these AEC’s and show that
under simplicity the notion has all the usual properties of first order non-forking
over complete types. Our approach generalizes the context of $\aleph_0$-stable homoge-
nous classes and excellent classes.

Our set of assumptions follow from disjoint amalgamation, existence of a prime
model over $\emptyset$, Löwenheim-Skolem number being $\omega$, $\text{LS}(\mathbb{K})$-tameness and a prop-
erty we call finite character. We also start the studies of these classes from the
$\aleph_0$-stable case. Stability in $\aleph_0$ and $\text{LS}(\mathbb{K})$-tameness can be replaced by categoric-
ity above the Hanf number. Finite character is the main novelty of this paper.
Almost all examples of AEC’s have this property and it allows us to use weak
types, as we call them, in place of Galois types.

1. Introduction

The context of abstract elementary classes, introduced by Shelah in [16], encom-
passes much of current model theory. An abstract elementary class consists of a
pair $(\mathbb{K}, \preceq_{\mathbb{K}})$, where $\mathbb{K}$ is a class of models in a fixed vocabulary $\tau$ and $\preceq_{\mathbb{K}}$ is a
notion of substructure extending the submodel relation, satisfying natural proper-
ties; mainly closure under isomorphism, closure under Tarski-Vaught chains, and
the existence of a cardinal $\text{LS}(\mathbb{K})$, called the Löwenheim-Skolem cardinal, such that
for any $A \subset A' \in \mathbb{K}$ there is $A'' \preceq_{\mathbb{K}} A'$ containing $A$ of size $|A| + \text{LS}(\mathbb{K})$. The relation $\preceq_{\mathbb{K}}$ yields a natural notion of $\mathbb{K}$-embedding $f : A \rightarrow B$, which are those
embeddings such that $f(A) \preceq_{\mathbb{K}} B$.

To study a particular class $\mathbb{K}$ of models in applications, there may be several
choices for $\preceq_{\mathbb{K}}$. Some of these may be more suitable, for example the class $\mathbb{K}$ may
have amalgamation under some notion of $\preceq_{\mathbb{K}}$ but not under others. The main idea
of this paper is to consider a natural property of $\preceq_{\mathbb{K}}$, called finite character, which

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is responsible for a good global behavior of \((\mathbb{K}, \preceq_{\mathbb{K}})\). We demonstrate this here by introducing a good independence relation, proving stability transfers, the existence of saturated models, and developing simplicity. Almost all examples of abstract elementary classes have finite character, but we also provide one example which doesn’t.

In order to explain what we mean by finite character, let us consider a class \(\mathbb{K}\) of models of a first order theory \(T\) with \(\preceq_{\mathbb{K}}\) taken as \(\preceq\), the elementary submodel relation. \((\text{Mod}(T), \preceq)\) is the prototypical example of an abstract elementary class. For \(\mathcal{A} \subset \mathcal{B}\) we have \(\mathcal{A} \preceq \mathcal{B}\) if and only if \(\text{tp}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}(\bar{a}/\emptyset, \mathcal{B})\), for each finite \(\bar{a} \in \mathcal{A}\). We call this property finite character; the relation \(\preceq\) depends on only finite amount of information at a time (we used to call this property ‘locality’ but changed it as it clashes with other notions, see Baldwin’s book [1]). The same property holds if we replace first order by any logic whose formulas have a finite number of free variables. Now consider an abstract elementary class \(\mathbb{K}\) with amalgamation and arbitrarily large models. This is essentially the context of Jónsson and Fraïssé, see [11]. In this context, we can define types semantically: For \(\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{K}\), and \(\bar{a}_1 \in \mathcal{A}_1, \bar{a}_2 \in \mathcal{A}_2\), we can define

\[ \text{tp}^g(\bar{a}_1/\emptyset, \mathcal{A}_1) = \text{tp}^g(\bar{a}_2/\emptyset, \mathcal{A}_2) \]

if there exist a model \(\mathcal{B} \in \mathbb{K}\) and \(\mathbb{K}\)-embeddings \(f_1 : \mathcal{A}_1 \rightarrow \mathcal{B}\) such that \(f_1(\bar{a}_1) = f_2(\bar{a}_2)\). This induces an equivalence relation under amalgamation, and we call the resulting equivalence class, written \(\text{tp}^g(\bar{a}/\emptyset, \mathcal{A})\), the Galois types of \(\bar{a}\) in \(\mathcal{A}\). Galois types generalize the usual notion of types, defined as sets of formulas. We say that \(\mathbb{K}\) has finite character, if \(\preceq_{\mathbb{K}}\) satisfies the condition above using \(\text{tp}^g\) instead of \(\text{tp}\).

In this paper, we study abstract elementary classes with finite character under slightly stronger amalgamation properties: we assume that \(\mathbb{K}\) has disjoint amalgamation and that there exists a prime model over the empty set. We consider the case where \(\mathbb{K}\) has countable downward Löwenheim-Skolem number and assume that \(\mathbb{K}\) is \(\aleph_0\)-stable. This setting generalises the \(\aleph_0\)-stable first order and homogeneous case, as well as excellent classes.

Because of amalgamation, \(\mathbb{K}\) has a well-behaved monster model \(\mathcal{M}\) in the sense that if \(f : \mathcal{A} \rightarrow \mathcal{B}\) is a \(\mathbb{K}\)-embedding and \(\mathcal{A}, \mathcal{B} \preceq_{\mathbb{K}} \mathcal{M}\) are small compared to the size of \(\mathcal{M}\) then \(f\) extends to an automorphism of \(\mathcal{M}\). But we can do better. Shelah’s Presentation Theorem states that there is a countable language \(\tau^*\) expanding \(\tau\), such that \(\mathbb{K}\) is the class of reducts to \(\tau\) of models of a first order theory \(T^*\) omitting a prescribed set of types \(\Gamma\). Moreover, if \(\mathcal{A} \preceq_{\tau^*} \mathcal{B}\) are models of \(T^*\) omitting all types in \(\Gamma\) then \(\mathcal{A} \upharpoonright \tau \preceq_{\mathbb{K}} \mathcal{B} \upharpoonright \tau\). Under our assumptions, we are able to show that there are arbitrarily large homogeneous models of \(T^*\) omitting all types in \(\Gamma\). Therefore, the monster model \(\mathcal{M}\) can be chosen as the reduct of a homogeneous
We obtain even better properties (see Theorem 2.18). This allows us to apply some of the methods of homogeneous model theory, in spite of the fact that we lose stability in the language $\tau^*$. We use this in the proof of symmetry and to have good control over indiscernible sequences.

The second main idea of this paper is to consider weak types. It is not difficult to see that the Galois types over the empty set we introduced earlier correspond to orbits of the automorphism group of the monster model $\mathcal{M}$. Working inside the monster model, we can generalize this idea and consider the Galois type of any finite sequence $\bar{a}$ over any set $B$, written $\text{tp}^g(\bar{a}/B)$, which is simply the orbit of $\bar{a}$ under the group of automorphisms of $\mathcal{M}$ fixing $B$ pointwise. Now given $\bar{a}, \bar{c} \in \mathcal{M}$ and $B \subseteq \mathcal{M}$, we say that

$$\text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{c}/B)$$

if and only if $\text{tp}^g(\bar{a}/B') = \text{tp}^g(\bar{c}/B')$ for each finite $B' \subseteq B$. This induces an equivalence relation and we call the equivalence class $\text{tp}^w(\bar{a}/B)$ the weak type of $\bar{a}$ over $B$. It follows immediately from the definition that if two weak types differ, they differ over a finite set. It is clear that Galois types and weak types coincide over finite sets, but we can also show that weak types and Galois types coincide over countable models.

We then consider a natural notion of splitting for weak types and are able to prove many of the usual properties over $\aleph_0$-saturated models: local character, finite character, transitivity, and stationarity. We can also prove the countable extension property. To find nonsplitting extensions to larger sets, we consider additional assumptions. For example, we obtain the full picture under categoricity (Corollary 4.21):

**Theorem 1.1.** Assume that $\mathbb{K}$ is categorical in some cardinal above the Hanf number. Then the independence relation based on splitting satisfies, in addition, the extension property and symmetry over $\aleph_0$-saturated models.

So if $\mathbb{K}$ is categorical in a big enough cardinal, then we have an independence relation which satisfies all the properties of forking in stable first order theories, provided we work over $\aleph_0$-saturated models. This provides further evidence for the validity of Shelah’s categoricity conjecture, see [18].

We get the same conclusion if we assume tameness instead of categoricity. Recall that $\mathbb{K}$ is tame if whenever $\text{tp}^g(\bar{a}/\mathcal{B}) \neq \text{tp}^g(\bar{b}/\mathcal{B})$ then there is a countable $\mathcal{A} \leq_{\mathbb{K}} \mathcal{B}$ such that $\text{tp}^g(\bar{a}/\mathcal{A}) \neq \text{tp}^g(\bar{b}/\mathcal{A})$. A consequence of tameness in our context is that $(\mathbb{K}, \leq_{\mathbb{K}})$ is stable in every infinite cardinal. It follows from the existence of such a well-behaved independence relation that there are weakly saturated models in every cardinality i.e., models which are saturated with respect to weak types. Since Galois types and weak types coincide over countable models, we deduce that there are
$\aleph_1$-saturated models in every cardinality. Furthermore, since tameness and finite character imply that Galois types and weak types coincide over all models, we obtain the following theorem (Theorem 4.15):

**Theorem 1.2.** If $\mathcal{K}$ is tame there is saturated model in every infinite cardinality.

The intuition is that weak types behave as the usual notion of syntactic types whereas Galois types correspond to the semantic aspect. This analogy is explored further in the second part of the paper. We introduce and study the notion of *Lascar strong types*. The equivalence of Lascar strong types is the finest invariant equivalence relation with a bounded number of classes. We introduce also Lascar splitting similarly to what is done in [7]. Lascar splitting is a version of strong splitting which depends on the good behavior of some indiscernible sequences, which we derive from the properties of the (expanded) monster model. We are able to show that the independence relation based on Lascar splitting is well-behaved and can define the $U$-rank. We define *simplicity*, a property that implies the independence relation to have all the first order properties of simple first order theories (but recall that we are in the $\aleph_0$-stable case here). This means that we have good independence relation over all sets, not just $\aleph_0$-saturated models. We finish with the following theorem (Corollary 6.18).

**Theorem 1.3.** If $\mathcal{K}$ is tame with finite $U$-rank then $\mathcal{K}$ is simple.

Many ideas appearing in this paper and originally from elementary model theory are due to Saharon Shelah, like splitting, strong splitting, independence and the ideas behind the proof of symmetry for splitting. See [14], [16], [17] and for independence see also [9]. The notion of Galois type over a model is due to Shelah, but the name Galois type was introduced in [3]. Also the notion of tameness was formulated under different name by Shelah, but it was more carefully studied and found useful in [20] and [4]. The disjoint amalgamation property for non-elementary classes is studied in [4]. The finite character property is based on ideas appearing in [6] and [5].

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2. Finitary abstract elementary classes

Let \( \tau \) be a fixed countable vocabulary. We define the basic context of this paper: abstract elementary classes (AEC), the amalgamation and joint embedding property, and prime model.

**Definition 2.1.** A class of \( \tau \)-structures \((K, \preceq_K)\) is an abstract elementary class if

1. Both \( K \) and the binary relation \( \preceq_K \) are closed under isomorphism.
2. If \( A \preceq_K B \), then \( A \) is a substructure of \( B \).
3. \( \preceq_K \) is a partial order on \( K \).
4. If \( \langle A_i : i < \delta \rangle \) is an \( \preceq_K \)-increasing chain, then
   - \( \bigcup_{i<\delta} A_i \in K \);
   - for each \( j < \delta \), \( A_j \preceq_K \bigcup_{i<\delta} A_i \).
   - if each \( A_i \preceq_K M \in K \), then \( \bigcup_{i<\delta} A_i \preceq_K M \).
5. If \( A, B, C \in K \), \( A \preceq_K C \), \( B \preceq_K C \) and \( A \subseteq B \) a subset, there is \( A' \in K \) such that \( B \subseteq A' \preceq_K A \) and \( |A'| = |B| + \text{LS}(K) \).

When \( A \preceq_K B \), we say that \( B \) is a \( K \)-extension of \( A \) and \( A \) is a \( K \)-submodel of \( B \).

**Definition 2.2.** If \( A, B \in K \) and \( f : A \to B \) an embedding such that \( f(A) \preceq_K B \), we say that \( f \) is a \( K \)-embedding.

**Definition 2.3** (Amalgamation properties). We say that \((K, \preceq_K)\) has the amalgamation property if it satisfies the following:

If \( A, B, C \in K \), \( A \preceq_K B \), \( A \preceq_K C \) and \( B \cap C = A \), there is \( D \in K \) and a map \( f : B \cup C \to D \) such that \( f \upharpoonright B \) and \( f \upharpoonright C \) are \( K \)-embeddings.

We say that \((K, \preceq_K)\) has the disjoint amalgamation property, if we in addition require that \( f(B) \cap f(C) = f(A) \).

**Definition 2.4** (Joint embedding). We say that \((K, \preceq_K)\) has the joint embedding property if for every \( A, B \in K \) there is \( C \in K \) and \( K \)-embeddings \( f : A \to C \) and \( g : B \to C \).

**Definition 2.5** (Prime model). We say that \((K, \preceq_K)\) has a prime model (or prime model over \( \emptyset \)) if there is \( A_P \in K \) such that for each \( A \in K \) there is a \( K \)-embedding \( f : A_P \to A \).

Clearly joint embedding property follows from amalgamation and prime model.

To define finite character we use the following concept of a Galois type of a tuple \( \bar{a} \) in a model \( \mathcal{A} \) over the empty set, written \( \text{tp}^\emptyset(\bar{a}/\emptyset, \mathcal{A}) \).
Definition 2.6 ($\mathcal{A}$-Galois type). For $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $\bar{a} \in \mathcal{A}$, $\bar{b} \in \mathcal{B}$ we define
\[
\text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^\mathbb{K}(\bar{b}/\emptyset, \mathcal{B})
\]
if there is $\mathcal{C} \in \mathbb{K}$ and $\mathbb{K}$-embeddings $f : \mathcal{A} \rightarrow \mathcal{C}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ such that $f(\bar{a}) = g(\bar{b})$.

With finite character, we can decide whether a model is a $\mathbb{K}$-submodel of another model by only looking at all finite parts of it.

Definition 2.7 (Finite character). We say that an AEC $(\mathbb{K}, \preceq)$ has finite character, if it satisfies the following: If, $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, $\mathcal{A} \subset \mathcal{B}$, and for each finite $\bar{a} \in \mathcal{A}$ we have that $\text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{B})$, then $\mathcal{A} \preceq \mathcal{B}$.

Clearly the converse always holds, i.e. if $\mathcal{A} \preceq \mathcal{B}$, then $\text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{B})$ for every finite $\bar{a} \in \mathcal{A}$. We will mostly use finite character when looking at mappings $f : \mathcal{A} \rightarrow \mathcal{B}$, where $\mathcal{A} \preceq \mathcal{B}$. This assumption gives a sufficient and necessary condition for the mapping to be a $\mathbb{K}$-embedding. This is a key property for our notion of type to be close enough to Galois types.

Lemma 2.8. Let $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, $\mathcal{A} \preceq \mathcal{B}$ and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping. Then the condition that for each $\bar{a} \in \mathcal{A}$
\[
(1) \quad \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{B}) = \text{tp}^\mathbb{K}(f(\bar{a})/\emptyset, \mathcal{B})
\]
is equivalent for $f$ being a $\mathbb{K}$-embedding.

Proof. Clearly from 1 it follows that $f$ is a $\tau$-embedding. By the definition of a $\mathbb{K}$-embedding and finite character we get that $f$ is a $\mathbb{K}$-embedding if and only if
\[
\text{tp}^\mathbb{K}(f(\bar{a})/\emptyset, f(\mathcal{A})) = \text{tp}^\mathbb{K}(f(\bar{a})/\emptyset, \mathcal{B})
\]
for each finite $\bar{a} \in \mathcal{A}$. Since $f : \mathcal{A} \rightarrow f(\mathcal{A})$ is an isomorphism, we have that $\text{tp}^\mathbb{K}(f(\bar{a})/\emptyset, f(\mathcal{A})) = \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{A})$ for every $\bar{a} \in \mathcal{A}$, and furthermore since $\mathcal{A} \preceq \mathcal{B}$,
\[
\text{tp}^\mathbb{K}(f(\bar{a})/\emptyset, f(\mathcal{A})) = \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^\mathbb{K}(\bar{a}/\emptyset, \mathcal{B}).
\]
Thus $f$ is a $\mathbb{K}$-embedding if and only if 1 holds. \qed

Finally we define our concept of finitary abstract elementary class.

Definition 2.9 (Finitary abstract elementary class). We say that an abstract elementary class $(\mathbb{K}, \preceq)$ is finitary, if it satisfies the following:

1. $\text{LS}^\mathbb{K}(\mathbb{K}) = \aleph_0$.
2. $(\mathbb{K}, \preceq)$ has arbitrarily large models,
3. $(\mathbb{K}, \preceq)$ has the disjoint amalgamation property,
4. $(\mathbb{K}, \preceq)$ has a prime model and...
(5) \((\mathbb{K}, \preceq_{\mathbb{K}})\) has finite character.

We could replace Definition 2.9(2) with the existence of a single proper extension of some model, since this together with disjoint amalgamation implies arbitrarily large models.

Amalgamation has been found essential in the study of abstract elementary classes. We want the models of the class to be 'algebraically closed'\(^1\) and thus require the disjoint amalgamation property. We use disjointness of the amalgamation in Lemma 2.16 to get the amalgamation property also for the class \(\mathbb{K}^*\) with extended vocabulary. Respectively, Prime model is needed to guarantee that the class \(\mathbb{K}^*\) has the joint embedding property. Prime model can be replaced with any assumption that guarantees this. We are also going to assume \(\aleph_0\)-stability after defining our notion of type in section 3, and thus we assume that the Löwenheim-Skolem number is \(\aleph_0\).

The main example of a finitary AEC is a so called excellent class defined in [15]. In the most general case an excellent class might not have a prime model, but the commonly considered atomic excellent classes do have it. Other properties of an excellent class are studied for example in [12]. Also an \(\aleph_0\)-stable homogeneous class, see [13] for the definition, is a finitary AEC, if we assume the existence of a prime model. We note that if we consider any \(\aleph_0\)-stable homogeneous class or any excellent class, we can take the class of \(\aleph_0\)-saturated models of this class, and the new class does have a prime model.

The next remark shows that if \(L\) is any logic with the property that any formula \(\phi \in L\) has only finitely many free variables, \(\mathbb{K}\) is a set of structures in the vocabulary of \(L\) and \(\preceq_{\mathbb{K}}\) is the elementary substructure relation in \(L\), then \((\mathbb{K}, \preceq_{\mathbb{K}})\) has finite character. Thus we gain various examples of finitary abstract elementary classes by studying models of a countable fragment of \(L_{\omega_1}\), if we in addition have properties (2)-(4) of Definition 2.9.

**Remark 2.10.** Assume that \(L\) is a logic such that every formula \(\phi \in L\) has only finitely many free variables. Let \(\mathbb{K}\) a set of structures of the vocabulary of \(L\), and \(\preceq_L\) the relation defined for \(\mathcal{A}, \mathcal{B} \in \mathbb{K}\) such that \(\mathcal{A} \preceq_L \mathcal{B}\) if \(\mathcal{A} \subset \mathcal{B}\) and for each \(\phi \in L\) and finite \(\bar{a} \in \mathcal{A}\), \(\mathcal{A} \models_L \phi(\bar{a})\) if and only if \(\mathcal{B} \models_L \phi(\bar{a})\). Then \((\mathbb{K}, \preceq_L)\) (although not necessarily an AEC) satisfies finite character.

**Proof.** Let \(\mathcal{A}, \mathcal{B} \in \mathbb{K}\) such that \(\mathcal{A} \subset \mathcal{B}\) and for every finite \(\bar{a} \in \mathcal{A}\) we have that \(\text{tp}^\mathcal{A}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^\mathcal{B}(\bar{a}/\emptyset, \mathcal{B})\). Let \(\phi(x_0,...,x_{n-1})\) be a formula in \(L\) and \(\bar{a} = (a_0,...,a_{n-1}) \in \mathcal{A}\). We want to show that \(\mathcal{A} \models_L \phi(\bar{a})\) if and only if \(\mathcal{B} \models_L \phi(\bar{a})\).

\(^1\)See bounded closure in section 5.1, especially Lemma 5.20. From disjoint amalgamation it follows that for a model \(\mathcal{A}\), \(\text{bcl}(\mathcal{A}) = \mathcal{A}\).
Let $B$ does not add any elements to the set. $A$ is a substructure of vocabulary $\{L, \lambda\}$, this class is also $(\mathbb{K}, \preceq_{\mathbb{K}})$-tame, for each uncountable $\lambda$.\footnote{See Definition 4.10} We divide the domain of a model $\mathcal{A}$ with two predicates $P_0$ and $P_1$, and then attach to every element in $P_0^{\mathcal{A}}$ a unique infinite subset of $P_1^{\mathcal{A}}$. When $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$, we demand that $\mathcal{A}$ is a substructure of $\mathcal{B}$ and if a subset $\subseteq P_1^{\mathcal{A}}$ named by $a \in P_0^{\mathcal{A}}$ is countable, $\mathcal{B}$ does not add any elements to the set.

**Example 2.11.** Let $R$ be a binary relation and $P_0, P_1$ unary. When $\mathcal{A}$ is a model of vocabulary $\{R, P_0, P_1\}$ and $a \in \mathcal{A}$, denote $R(a, \mathcal{A}) = \{b \in \mathcal{A} : (a, b) \in R^{\mathcal{A}}\}$. Let $\mathbb{K}$ be those models $\mathcal{A}$ of this vocabulary that satisfy

1. For all $a \in P_0^{\mathcal{A}}$, $R(a, \mathcal{A})$ is infinite.
2. If $R(a, b)$, then $P_0(a)$ and $P_1(b)$.
3. $R(a, b)$ and $R(c, b)$ imply $a = c$.
4. $P_0^{\mathcal{A}} \cup P_1^{\mathcal{A}} = \mathcal{A}$ and $P_0^{\mathcal{A}} \cap P_1^{\mathcal{A}} = \emptyset$.

Then define for $\mathcal{A}, \mathcal{B} \in \mathbb{K}$, that $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ if

1. $\mathcal{A}$ is a substructure of $\mathcal{B}$,
2. if $a \in P_0^{\mathcal{A}}$ and $R(a, \mathcal{A})$ is countable, then $R(a, \mathcal{B}) \subseteq \mathcal{A}$ and
3. if $b \in P_1^{\mathcal{A}}$, $a \in \mathcal{B}$ and $R(a, b)$ holds, then $a$ is already in $\mathcal{A}$.

This example does not have finite character, since we may take two countably infinite sets $A$ and $B$, where $A \subseteq B$ and there is exactly one element $b \in B \setminus A$. Let $a_0$ be an element in $A$. If we take $P_0^{\mathcal{A}} = P_0^{\mathcal{B}} = \{a_0\}$, $P_1^{\mathcal{A}} = A \setminus \{a_0\}$, $P_1^{\mathcal{B}} = B \setminus \{a_0\}$, $(a_1, a_2) \in R^{\mathcal{A}}$ if and only if $a_1 = a_0$ and $a_2 \neq a_0$, and similarly for $R^{\mathcal{B}}$, we get that $\mathcal{A} = (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, R^{\mathcal{A}})$ and $\mathcal{B} = (B, P_0^{\mathcal{B}}, P_1^{\mathcal{B}}, R^{\mathcal{B}})$ are in $\mathbb{K}$. Since $R(a_0, \mathcal{B}) = A \cup \{b\}$ is not in $\mathcal{A}$, we have that $\mathcal{A} \nsubseteq_{\mathbb{K}} \mathcal{B}$. Although for every finite $\bar{a} \in \mathcal{A}$ we can $\mathbb{K}$-embed $b$ into $\mathcal{A}$ and fix $\bar{a}^{-} a_0$. Thus $tp^{\mathcal{A}}(\bar{a}/\emptyset, \mathcal{A}) = tp^{\mathcal{A}}(\bar{a}/\emptyset, \mathcal{B})$ for every finite $\bar{a} \in \mathcal{A}$. This contradicts finite character. Other details of the example are left to the reader.

From now on we assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary abstract elementary class. We recall the assumptions needed in the beginning of each section and again in every theorem.
### Assumption 2.12
We assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a finitary abstract elementary class.

#### 2.1. Extended vocabulary \(\tau^*\) and the monster model.
This section is based on ideas due to Shelah. In [15] he shows that an abstract elementary class is actually a so-called PC-class. Similarly, we introduce an extended vocabulary with some Skolem-functions, and show a version of Shelah’s representation theorem. Using disjoint amalgamation and prime model, we will get a monster model with very good homogeneity properties with respect to types in the extended vocabulary. This will be used first time when proving symmetry for splitting in section 4, and then again when considering strongly indiscernible sequences in section 5.

Amalgamation, joint embedding, arbitrarily large models and finite character give the monster model another homogeneity property we call \(\mathbb{K}\)-homogeneity.

#### Definition 2.13
For \(n, k < \omega\), let \(F_n^k\) be a \(k\)-ary function symbol, \(\tau^* = \tau \cup \{F_n^k : n, k < \omega\}\) and \(\mathbb{K}^*\) be the class of all \(\tau^*\)-structures such that for \(\mathcal{A} \in \mathbb{K}^*\):

1. \(\mathcal{A} \upharpoonright \tau \in \mathbb{K}\),
2. For all \(\bar{a} \in \mathcal{A}\), \(\mathcal{A}_{\bar{a}} = \{((F_n^k)(\bar{a}))_{\mathcal{A}} : n < \omega\}\) is such that
   a. \(\mathcal{A}_{\bar{a}} \in \mathbb{K}\) and \(\mathcal{A}_{\bar{a}} \preceq_{\mathbb{K}} \mathcal{A} \upharpoonright \tau\),
   b. if \(\bar{b} \preceq \bar{a}\) then \(\bar{b} \in \mathcal{A}_{\bar{a}} \subseteq \mathcal{A}_{\bar{a}}^3\).
   c. Let \((a_i)_{i<\omega}\) be a fixed ordering on \(\mathcal{A}_{\mathcal{P}}\). The mapping \(f : \mathcal{A}_{\mathcal{P}} \to \mathcal{A}\), where \(f(a_i) = (F_i^0)_{\mathcal{A}}\), is a \(\mathbb{K}\)-embedding.
   d. For all \(0 < k < \omega\), \(i_0, ..., i_{k-1}\) and \(n < \omega\), \(F_n^k(F_{i_0}^0, ..., F_{i_{k-1}}^0) = F_n^0\).

Items (2)(c) and (2)(d) ensure that two models \(\mathcal{A}, \mathcal{B} \in \mathbb{K}^*\) satisfy the same atomic \(\tau^*\)-sentences, and thus \(((F_i^0)_{\mathcal{A}})_{i<\omega}\) is \(\tau^*\)-isomorphic to \(((F_i^0)_{\mathcal{B}})_{i<\omega}\). Also from (2)(d) it follows that the domain of a model \(\mathcal{A}_{\bar{a}}\) is the set \(\{(F_i^0)_{\mathcal{A}} : i < \omega\}\), whenever \(\bar{a}\) is in the set \(\{(F_i^0)_{\mathcal{A}} : i < \omega\}\).

By induction on the size of \(B\) we can prove the following, see [2] for the proof.

#### Lemma 2.14
If \(\mathcal{A} \in \mathbb{K}^*\) and \(B \subseteq \mathcal{A}\) a subset such that \(B\) is closed under functions \(F_n^k\), then \(B \upharpoonright \tau \in \mathbb{K}\) and \(B \upharpoonright \tau \preceq_{\mathbb{K}} \mathcal{A} \upharpoonright \tau\).

Note that if \(\mathcal{A}, \mathcal{B} \in \mathbb{K}^*\) and \(f : \mathcal{A} \to \mathcal{B}\) is a \(\tau^*\)-embedding, then \(f : \mathcal{A} \upharpoonright \tau \to \mathcal{B} \upharpoontright \tau\) is a \(\mathbb{K}\)-embedding. This follows from Lemma 2.14, since an image of a model in an embedding is closed under functions.

Of course from Lemma 2.14 it follows that if \(\mathcal{B}\) is a \(\tau^*\)-submodel of \(\mathcal{A} \in \mathbb{K}^*\), then also \(\mathcal{B} \preceq_{\mathbb{K}} \mathcal{A} \upharpoonright \tau\). Thus the properties (1)-(5) of definition 2.1 hold for \(\mathbb{K}^*\) where \(\preceq_{\mathbb{K}}\) is replaced with the \(\tau^*\)-submodel relation.

\(^3\)Here \(\bar{b} \preceq \bar{a}\) means that \(\ell(\bar{b}) \leq \ell(\bar{a})\) and the members of the tuple \(\bar{b}\) are contained in the set of members of \(\bar{a}\), i.e. when \(\bar{b} = (b_0, ..., b_k)\) and \(\bar{a} = (a_0, ..., a_n)\), \(\{b_0, ..., b_k\} \subseteq \{a_0, ..., a_n\}\).
Lemma 2.15. For every $\mathcal{A} \in \mathbb{K}$ there is $\mathcal{A}^* \in \mathbb{K}^*$ such that $\mathcal{A}^* \models \tau = \mathcal{A}$.

Proof. We have to define functions $(F^i_n)^{\mathcal{A}}$ so that they satisfy the conditions in Definition 2.13. We do that by defining functions by induction on $\ell(\bar{a})$, and for all $\bar{a} \in \mathcal{A}$ of the same length simultaneously, with the exception that all functions on the constants $F^0_i$ are determined by the definition. Otherwise we notice that $\mathcal{A}_b$ of Definition 2.13 need not to depend on the ordering of $\bar{a}$, thus we let $(F^i_n(\bar{a}))^{\mathcal{A}} = (F^i_n(\bar{a}))^{\mathcal{A}}(\beta(\bar{a}))$, whenever $\beta : \bar{a} \to \bar{a}$ is a bijection. Also if the elements of $\bar{a}$ are already contained in some shorter sequence $\bar{a}'$, we let $\mathcal{A}_b$ be equal to $\mathcal{A}_{\bar{a}'}$.

1° First define constants $(F^0_i)^{\mathcal{A}}_{i < \omega}$. Let $f$ be a $\mathbb{K}$-embedding of the prime model $\mathcal{A}_{i<\omega}$ into $\mathcal{A}$ and $(b_i)_{i<\omega}$ be the fixed ordering on $\mathcal{A}_{i<\omega}$. We define $(F^0)^{\mathcal{A}} = f(b_i)$ for each $i < \omega$. Now also all functions on the constants are determined by (2)(d) of Definition 2.13.

2° Assume we have defined $(F^i_n(\bar{a}))^{\mathcal{A}}$ for each $\bar{a}$ of length less or equal to $n$ and for all $i < \omega$. Then define functions for each $\bar{b} \in \mathcal{A}^{n+1}$. We want to check that permutation does not affect the choice of $\mathcal{A}_b$, thus we order $\mathcal{A}^{n+1}$ and compare $\bar{b} \in \mathcal{A}^{n+1}$ with the previous ones. Let $\bar{b} \in \mathcal{A}^{n+1}$ and assume we have defined functions for the previous $\bar{b}' \in \mathcal{A}^{n+1}$. If the elements of the sequence $\bar{b}$ are already contained in some shorter sequence $\bar{b}'$, $\bar{b}$ is a permutation of some previous $\bar{b}' \in \mathcal{A}^{n+1}$ or $\bar{b} = \bar{b}' \in \{(F^0)^{\mathcal{A}} : i < \omega\}$, let $((F^{n+1})^{\mathcal{A}}(\bar{b})) = ((F^{n+1})^{\mathcal{A}}(\bar{b}'))$ for each $i < \omega$. Otherwise we do the following. Since $LS(\mathbb{K}) = \omega$, there is $\mathcal{A}_b \in \mathbb{K}$ such that $|\mathcal{A}_b| \leq \omega$, $\mathcal{A}_b \subseteq \mathcal{A}$ and $F \subset \mathcal{A}_b$, where $F$ is the countable set

$$F = \{(F^i_n(\bar{a}))^{\mathcal{A}} : \bar{a} \subset \bar{b}, \ell(\bar{a}) < \ell(\bar{b}), i < \omega\} \cup \{\bar{b}\}.$$ 

We let $((F^i_n(\bar{a}))^{\mathcal{A}}(\bar{b}))_{i<\omega}$ enumerate $\mathcal{A}_b$. When we have defined functions for each $\bar{b} \in \mathcal{A}^{n+1}$, we see that $\mathcal{A}_b \subset \mathcal{A}_b$ whenever $\bar{a} \subset \bar{b}$.

Lemma 2.16 ($\mathbb{K}^*$-amalgamation). If $\mathcal{A}, \mathcal{B} \in \mathbb{K}^*$ such that for each $\bar{c} \in \mathcal{A} \cap \mathcal{B}$ and atomic $\tau^*$-sentence $\psi$,

$$\mathcal{A} \models \psi(\bar{c}) \Leftrightarrow \mathcal{B} \models \psi(\bar{c}),$$

then there is $\mathcal{C} \in \mathbb{K}^*$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{C}$ such that $f \models \mathcal{A}$ and $f \models \mathcal{B}$ are $\tau^*$-embeddings.

Proof. Denote $(\mathcal{A} \cap \mathcal{B})^{\mathcal{A}}$ to be the closure of $(\mathcal{A} \cap \mathcal{B})$ under functions $(F^k_n)^{\mathcal{A}}$, $k, n \in \omega$, and $(\mathcal{A} \cap \mathcal{B})^{\mathcal{B}}$ respectively. Now by the assumption $(\mathcal{A} \cap \mathcal{B})^{\mathcal{A}}$ and $(\mathcal{A} \cap \mathcal{B})^{\mathcal{B}}$ are isomorphic over $\mathcal{A} \cap \mathcal{B}$ and by Lemma 2.14 belong to $\mathbb{K}^*$. Let $h : (\mathcal{A} \cap \mathcal{B})^{\mathcal{A}} \to (\mathcal{A} \cap \mathcal{B})^{\mathcal{B}}$ be an isomorphism such that $h \models (\mathcal{A} \cap \mathcal{B}) = \text{id}_{(\mathcal{A} \cap \mathcal{B})}$. We find $\mathcal{B}'$ and an isomorphism $h' : \mathcal{B} \to \mathcal{B}'$ such that $h' \circ h \models (\mathcal{A} \cap \mathcal{B}) = \text{id}_{(\mathcal{A} \cap \mathcal{B})}$ and $\mathcal{A} \cap \mathcal{B}' = (\mathcal{A} \cap \mathcal{B})^{\mathcal{A}} = (h'((\mathcal{A} \cap \mathcal{B})^{\mathcal{B}})) = (\mathcal{A} \cap \mathcal{B}')^{\mathcal{B}}$. From Lemma
Theorem 2.18. Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is a finitary AEC. Let \(\mu\) be a cardinal. There is \(\mathbb{M}^* \in \mathbb{K}^*\) such that:

1. **\(\mu\)-Universality:** \(\mathbb{M}^*\) is \(\mu\)-universal, that is for each \(\mathcal{A} \in \mathbb{K}^*\), \(|\mathcal{A}| < \mu\), there is a \(\tau^*\)-embedding \(f : \mathcal{A} \to \mathbb{M}^*\).

2. **\(\mu\)-Homogeneity:** When \((a_i)_{i<\alpha}, (b_i)_{i<\alpha} \subseteq M^\ast\), \(\alpha < \mu\), and for each \(i_0, \ldots, i_n < \alpha\) and \(\psi\) atomic \(\tau^*\)-formula,

\[
\mathbb{M}^* \models \psi(a_{i_0}, \ldots, a_{i_n}) \iff \mathbb{M}^* \models \psi(b_{i_0}, \ldots, b_{i_n}),
\]

there is \(f \in \text{Aut}(\mathbb{M}^*)\) such that \(f(a_i) = b_i\) for each \(i < \alpha\).

3. **\(\mathbb{K}\)-homogeneity** For all \(\mathcal{A} \preceq_\mathbb{K} \mathbb{M}^* \upharpoonright \tau\) such that \(|\mathcal{A}| < \mu\) and mappings \(f : \mathcal{A} \to \mathbb{M}^*\) such that for all finite tuples \(\bar{a} \in \mathcal{A}\)

\[
\text{tp}(\bar{a}/\emptyset, \mathbb{M}^* \upharpoonright \tau) = \text{tp}(f(\bar{a})/\emptyset, \mathbb{M}^* \upharpoonright \tau),
\]

there is \(g \in \text{Aut}(\mathbb{M}^* \upharpoonright \tau)\) extending \(f\).
We denote $\mathcal{M} = \mathcal{M}^* \upharpoonright \tau$. In the case of AEC with amalgamation and joint embedding, $\mathbb{K}$-homogeneity is the ability to extend $\mathbb{K}$-embeddings $f : \mathcal{A} \to \mathcal{M}$, when $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{M}$, to automorphisms of $\mathcal{M}$. By finite character, when $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{M}$, a mapping $f : \mathcal{A} \to \mathcal{M}$ is a $\mathbb{K}$-embedding if and only if it preserves types of finite tuples as stated in (3). This formulation of item (3) is the only place in section 2.1 where we use finite character. The first ‘real’ use of the property will be in Lemma 3.4.

The next remark follows clearly from the homogeneity of $\mathcal{M}^*$.

**Remark 2.19.** Let $A \subset \mathcal{M}^*$, $|A| < \mu$, be a subset and $(c_a)_{a \in A}$ a set of new constants. Then let $\mathcal{M}^*_A = (\mathcal{M}^*, c_a)_{a \in A}$ be the model where each constant $c_a$ is interpreted as $a$. The following are equivalent for all subsets $B = (b_i)_{i \in I}$ and $C = (c_i)_{i \in I}$ of $\mathcal{M}^*$, when $|I| < \mu$.

1. For all first order formulas $\phi$ of vocabulary $\tau^* \cup \{c_a : a \in A\}$, $n < \omega$ and indexes $i_0, ..., i_n \in I$, $\mathcal{M}^*_A \models \phi(b_{i_0}, ..., b_{i_n})$ if and only if $\mathcal{M}^*_A \models \phi(c_{i_0}, ..., c_{i_n})$.
2. For all atomic formulas $\phi$ of vocabulary $\tau^* \cup \{c_a : a \in A\}$, $n < \omega$ and indexes $i_0, ..., i_n \in I$, $\mathcal{M}^*_A \models \phi(b_{i_0}, ..., b_{i_n})$ if and only if $\mathcal{M}^*_A \models \phi(c_{i_0}, ..., c_{i_n})$.
3. For all $n < \omega$ and indexes $i_0, ..., i_n \in I$ there is an automorphism $f$ of $\mathcal{M}^*$ such that $f(b_{i_k}) = c_{i_k}$ for $0 \leq k \leq n$ and $f \upharpoonright A = \text{id}_A$.
4. There is an automorphism $f$ of $\mathcal{M}^*$ such that $f(b_i) = c_i$ for each $i \in I$ and $f \upharpoonright A = \text{id}_A$.

We define also $\tau^*$-type and $\tau^*$-order-indiscernible here.

**Definition 2.20 (\(\tau^*\)-type).** Let $B = (b_i)_{i \in I}$ and $C = (c_i)_{i \in I}$ be subsets of $\mathcal{M}$. We write

\[ \text{tp}^*(B/A) = \text{tp}^*(C/A) \]

if one (and all) of the conditions (1), (2), (3) and (4) of Remark 2.19 hold for $B$ and $C$.

**Definition 2.21 (\(\tau^*\)-order-indiscernible).** Let $(I, <)$ be a linear ordering. We say that a sequence $(\bar{a}_i)_{i \in I}$ is $n$-indiscernible over $A$ if for each $i_0 < ... < i_{n-1} \in I$ and $j_0 < ... < j_{n-1} \in I$

\[ \text{tp}^*(\bar{a}_{i_0}, ..., \bar{a}_{i_{n-1}}/A) = \text{tp}^*(\bar{a}_{j_0}, ..., \bar{a}_{j_{n-1}}/A). \]

We say that the sequence is $\tau^*$-order-indiscernible if it is $n$-indiscernible for each $n < \omega$.

In the following lemmas we recall two properties of homogeneous classes. These are results to produce and extend $\tau^*$-order indiscernible sequences, and will be used later in sections 4 and 5.
Lemma 2.22. There exists a cardinal $\lambda$ such that for each countable $A$ and a set $\{\bar{a}_i : i < \lambda\}$ there exists a sequence $(\bar{b}_i)_{i<\omega}$ such that it is $\tau^*$-order-indiscernible over $A$ and for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that
$$tp^*(\bar{b}_0, \ldots, \bar{b}_n/A) = tp^*(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/A).$$

Lemma 2.23. Let $(\bar{a})_{i<\omega}$ be a $\tau^*$-order-indiscernible sequence over $A$ and $(I, <')$ a linear ordering. There are tuples $(\bar{c}_i)_{i \in (I, <')}$ in $\mathcal{M}$ such that for each $n < \omega$ and $i_0 <' \ldots <' i_n$
$$tp^*(\bar{c}_{i_0}, \ldots, \bar{c}_{i_n}/A) = tp^*(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/A).$$

We call the cardinal $\lambda$ of Lemma 2.22 the Hanf number, written $\mathbb{H}$. We can calculate that $\mathbb{H} = \sum_{2^{\aleph_0}}^+ \mathbb{H}$.

3. Weak types

From now on we will assume that everything takes place in a large enough monster model $\mathcal{M}$, which is the restriction of the homogeneous monster model $\mathcal{M}^*$ to the vocabulary $\tau$. If we say that $\mathcal{A}$ is a model, we mean that $\mathcal{A} \in \mathbb{K}$ and $\mathcal{A} \equiv \mathcal{M}$. We also assume that we can apply the homogeneity and universality properties of Theorem 2.18 to every model and set under discussion. When $A$ is a set, we denote $\text{Aut}(\mathcal{M}/A) = \{ f \in \text{Aut}(\mathcal{M}) : f \upharpoonright A = \text{id}_A \}$.

We don’t use the extended vocabulary $\tau^*$ in this section, and thus the disjointness of the amalgamation is not needed here and prime model can be replaced by joint embedding. All the results in this section hold for an AEC with amalgamation, joint embedding, arbitrarily large models, countable Löwenheim-Skolem number and finite character.

We use the standard notion of a Galois type, except that we define it also over arbitrary sets, not only models.

Definition 3.1 (Galois type). We write $tp^g(\bar{a}/A) = tp^g(\bar{b}/A)$ if there is $f \in \text{Aut}(\mathcal{M}/A)$ such that $f(\bar{a}) = \bar{b}$.

The notion of Galois type does not necessarily have finite character, i.e. it is possible that $tp^g(\bar{a}/A) \neq tp^g(\bar{b}/A)$ although $tp^g(\bar{a}/B) = tp^g(\bar{b}/B)$ for every finite $B \subset A$. Proof of the following remark is left to the reader.

Remark 3.2. For all $\bar{a}$ and $\bar{b}$, $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$ if and only if $tp^g(\bar{a}/\emptyset, \mathcal{M}) = tp^g(\bar{b}/\emptyset, \mathcal{M})$.

In the following we define our notion of type with built-in finite character. Our notion of type is called a weak type, since equality of weak types is a weaker notion
than equality of Galois types. If two tuples \( \bar{a} \) and \( \bar{b} \) in the monster model have same weak type over some model \( \mathcal{A} \), that does not guarantee that there is an automorphism sending \( \bar{a} \) to \( \bar{b} \) and fixing \( \mathcal{A} \) pointwise. In section 4 we will see that under tameness and \( \aleph_0 \)-stability we will gain also the automorphism. Only \( \aleph_0 \)-stability is needed to find the automorphism when \( \mathcal{A} \) is countable, see Theorem 3.12. Over finite sets, equality of weak types and equality of Galois types always coincide.

We also want to define a dependence relation, where dependencies are between finite tuples. Therefore we need a concept of type, which talks only about finite sets.

**Definition 3.3 (Weak type).** Let \( \mathcal{A} \in \mathbb{K} \) and \( \bar{a}, \bar{b}, A \) be in \( \mathcal{A} \). We write \( \text{tp}^w(\bar{a}/A, \mathcal{A}) = \text{tp}^w(\bar{b}/A, \mathcal{A}) \) if \( \text{tp}^g(\bar{a} \downharpoonright \bar{c}/\emptyset, \mathcal{A}) = \text{tp}^g(\bar{b} \downharpoonright \bar{c}/\emptyset, \mathcal{A}) \) for every finite \( \bar{c} \in A \).

When we work inside the monster model \( \mathcal{M} \), we just write \( \text{tp}^w(\bar{a}/A) \) instead of \( \text{tp}^w(\bar{a}/\mathcal{M}) \). In the monster model \( \text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{b}/A) \) if and only if \( \text{tp}^g(\bar{a}/B) = \text{tp}^g(\bar{b}/B) \) for each finite \( B \subset A \).

In the following lemma we consider \( \aleph_0 \)-unions of Galois types. Finite character is used to fill the gap between the corresponding result in homogeneous classes and what can be done in general AEC with amalgamation and joint embedding. The sets \( A_n \) in the lemma are not necessarily models, and thus without finite character we don’t know whether the union \( \bigcup_{n<\omega} F_{0,n} \downdownarrows A_n \) is a \( \mathbb{K} \)-embedding. The requirement that \( \bigcup_{n<\omega} A_n \) is a model can not be removed, see e.g. [21], where a counter example is given.

**Lemma 3.4.** Let \( (A_n : n < \omega) \) be an increasing sequence of sets such that \( \bigcup_{n<\omega} A_n \) is a model in \( \mathbb{K} \). Let \( (\bar{b}_n)_{n<\omega} \) be finite sequences, \( \ell(\bar{b}_n) = n \), such that

\[
\text{tp}^g(\bar{b}_m \upharpoonright n/A_n) = \text{tp}^g(\bar{b}_n/A_n), \quad \text{for each } n < m < \omega.
\]

Then there exists an increasing sequence \( (\bar{c}_n : n < \omega), \) i.e. \( \bar{c}_n \downdownarrows m = \bar{c}_m \) for \( m < n < \omega \), such that

\[
\text{tp}^g(\bar{c}_n/A_n) = \text{tp}^g(\bar{b}_n/A_n), \quad \text{for each } i < \omega.
\]

**Proof.** Construct \( F_{i,j} \in \text{Aut}(\mathcal{M}/A_i) \) for \( i < j \) such that

1. \( F_{i,j}(\bar{b}_j \upharpoonright i) = \bar{b}_i \)
2. \( F_{i,j} = F_{i,k} \circ F_{k,j}, \) if \( i < k < j \).

Suppose that \( F_{i,j} \) have been constructed with \( i < j \leq n \). Then by assumption there exists \( F_{n,n+1} \in \text{Aut}(\mathcal{M}/A_n) \) such that \( F_{n,n+1}(\bar{b}_{n+1} \upharpoonright n) = \bar{b}_n \). Then define \( F_{i,n+1} = F_{i,n} \circ F_{n,n+1}, \) for each \( i < n \).
Denote \( \mathcal{A} = \bigcup_{n<\omega} A_n \in \mathbb{K} \) and \( g = \bigcup_{n<\omega} F_{0,n} \upharpoonright A_n \). Since \( \mathcal{A} \) is a model and for each finite \( \bar{a} \in \mathcal{A} \),

\[
\text{tp}^g(\bar{a}/\emptyset) = \text{tp}^g(g(\bar{a})/\emptyset),
\]

the mapping \( g : \mathcal{A} \to \mathfrak{M} \) extends to an automorphism \( G \) of \( \mathfrak{M} \). Now let \( \bar{c}_n = G^{-1} \circ F_{0,n}(\bar{b}_n) \), for each \( n < \omega \). Notice that \( \bar{c}_n \) is an increasing sequence by (2) since

\[
F_{0,n}(\bar{b}_n) = F_{0,n}(F_{n,n+1}(b_{n+1} \upharpoonright n)) = F_{0,n+1}(b_{n+1} \upharpoonright n).
\]

And finally \( t(\bar{c}_n/A_n) = t(\bar{b}_n/A_n) \), since \( G^{-1} \circ F_{0,n} \) is the identity on \( A_n \).

We mention separately this weaker version of the previous lemma.

**Corollary 3.5.** Let \( (A_n : n < \omega) \) be an increasing sequence of sets such that \( \bigcup_{n<\omega} A_n \) is a model in \( \mathbb{K} \). Let \( (\bar{b}_n)_{n<\omega} \) be finite sequences of the same length, such that

\[
\text{tp}^g(\bar{b}_m/A_n) = \text{tp}^g(\bar{b}_n/A_n), \quad \text{for each } n < m < \omega.
\]

Then there exists a tuple \( \bar{a} \) such that

\[
\text{tp}^g(\bar{a}/A_n) = \text{tp}^g(\bar{b}_n/A_n), \quad \text{for each } n < \omega.
\]

Now we introduce a new assumption for \( (\mathbb{K}, \preceq_{\mathbb{K}}) \). From now on we will assume that \( (\mathbb{K}, \preceq_{\mathbb{K}}) \) is an \( \aleph_0 \)-stable finitary abstract elementary class.

**Assumption 3.6** (\( \aleph_0 \)-stability). If \( A \subset \mathcal{A} \in \mathbb{K} \), \( A \) is countable and \( \bar{a}_i \in \mathcal{A} \) for \( i < \omega_1 \), then for some \( i < j < \omega_1 \), \( \text{tp}^w(\bar{a}_i/A, \mathcal{A}) = \text{tp}^w(\bar{a}_j/A, \mathcal{A}) \).

The standard notion of \( \aleph_0 \)-stability is that there are at most countably many Galois types over a countable model, and we call this standard notion \( \aleph_0 \)-Galois-stability. Our \( \aleph_0 \)-stability clearly follows from \( \aleph_0 \)-Galois-stability, and Theorem 3.12 will show that in our context the two notions agree.

**Definition 3.7.** We say that a submodel \( A \subset \mathfrak{M} \) is \( \aleph_0 \)-saturated if for each \( \bar{a} \in \mathfrak{M} \) and finite \( B \subset A \) there is \( \bar{b} \in A \) such that \( \text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B) \).

By \( \aleph_0 \)-stability, there are countable \( \aleph_0 \)-saturated models. Using finite character, we can show that \( \aleph_0 \)-saturated substructures of \( \mathfrak{M} \) are \( \mathbb{K} \)-substructures.

**Lemma 3.8.** Assume \( A \) is a countable set and the following holds: for each finite \( A_0 \subset A \) and \( \bar{b} \) there is \( \bar{d} \in A \) such that \( \text{tp}^w(\bar{a}/A_0) = \text{tp}^w(\bar{d}/A_0) \). Then \( A \preceq_{\mathbb{K}} \mathfrak{M} \).

**Proof.** Let \( \mathcal{B} \preceq_{\mathbb{K}} \mathfrak{M} \) be countable and \( \aleph_0 \)-saturated. Let \( A = \{a_n : n < \omega\} \) and \( \mathcal{B} = \{b_n : n < \omega\} \). Define inductively sets \( A_n \) and \( B_n \) and automorphisms \( f_n \) such that for each \( n < \omega \)

1. \( f_n(A_n) = B_n \),
2. \( \{a_0, \ldots, a_{n-1}\} \subset A_n \subset A \) and \( \{b_0, \ldots, b_{n-1}\} \subset B_n \subset \mathcal{B} \).
Let $f_0 = Id \upharpoonright \mathcal{M}$, $A_0 = \emptyset$ and $B_0 = \emptyset$. Then assume we have defined $f_m$, $A_m$ and $B_m$ for $m \leq n$.

By $\aleph_0$-saturation there exists $g \in \text{Aut}(\mathcal{M})$ such that $g(f_n(a_n)) \in \mathcal{B}$ and $g \upharpoonright B_n = \text{id}_{B_n}$. Then by the assumption there exists $h \in \text{Aut}(\mathcal{M})$ such that $h \upharpoonright A_n \cup \{a_n\}$ is the identity and $h(f_n^{-1} \circ g^{-1}(b_n)) \in A$. Define

$$f_{n+1} = g \circ f_n \circ h^{-1},$$

$$A_{n+1} = A_n \cup \{a_n\} \cup \{(h \circ f_n^{-1} \circ g^{-1})(b_n)\}$$

and

$$B_{n+1} = B_n \cup \{(g \circ f_n)(a_n)\} \cup \{b_n\}.$$ 

Then we get that $f_{n+1}(A_{n+1}) = B_{n+1}$.

Finally $f = \bigcup_{n<\omega} (f_n)^{-1} \upharpoonright B_n : \mathcal{B} \to \mathcal{M}$ is a $\mathbb{K}$-embedding, since it satisfies the property 1 of Lemma 2.8. Thus $f(\mathcal{B}) = A \preceq_{\mathbb{K}} \mathcal{M}$. \qedhere

With a similar back-and-forth construction, taking $A_0 = B_0 = E$, we can prove that two countable $\aleph_0$-saturated models containing a finite set $E$ are isomorphic over $E$, and thus by $\mathbb{K}$-homogeneity of $\mathcal{M}$, also automorphic over $E$. If both $\mathcal{A}$ and $\mathcal{B}$ are models, finite character is not needed for this. In particular, two countable $\aleph_0$-saturated models are isomorphic.

Now we introduce some tools for proving theorem 3.12.

**Definition 3.9** (Weakly isolated type). We say that a type $tp^w(\bar{b}/\mathcal{A} \cup \bar{a})$ is weakly isolated over finite $\mathcal{A} \cup \bar{a}$, if whenever $\bar{d}$ realizes $tp^w(\bar{b}/\mathcal{A} \cup \bar{a})$, then $\bar{d}$ realizes $tp^w(\bar{b}/\mathcal{A} \cup \bar{a})$.

**Lemma 3.10.** Assume $\mathcal{A}$ is a countable model, $A$ a finite subset of $\mathcal{A}$ and $\bar{a}$ be given. Then for each $\bar{b}$ there are $\bar{c}$ and a finite $\mathcal{A}' \subset \mathcal{A}$ such that

i) $\bar{c}$ realizes $tp^w(\bar{b}/A \cup \bar{a})$ and

ii) $tp^w(\bar{c}/\mathcal{A}' \cup \bar{a})$ is weakly isolated over $\mathcal{A}' \cup \bar{a}$.

**Proof.** Write $\mathcal{A} = A \cup \{a_n : n < \omega\}$. Suppose, for a contradiction, that the conclusion fails. We will construct a tree of types to contradict $\aleph_0$-stability. We construct an increasing sequence of finite sets $(A_n : n < \omega)$ and sequences $c_\eta$, for $\eta \in \omega^\omega$, such that

1. $c_0 = \bar{b}$, $A_0 = A$.
2. $a_n \in A_{n+1}$, $A_n \subseteq A_{n+1} \subseteq \mathcal{A}$.
3. If $\eta < \nu$ then $c_\nu$ realizes $tp^w(\bar{c}_\eta/A_\ell(\eta) \cup \bar{a})$.
4. $tp^w(\bar{c}_\eta/A_{n+1} \cup \bar{a}) \neq tp^w(\bar{c}_{\eta-1}/A_{n+1} \cup \bar{a})$, where $n = \ell(\eta)$.
We do this by induction on \( n = \ell(\eta) \). For \( n = 0 \), this is easy. Now \( t(c/\mathfrak{A}_1 \cup \bar{a}) \) cannot weakly isolate \( \text{tp}^w(c/\mathfrak{A} \cup \bar{a}) \) by assumption, so there are \( c_0 \), \( c_1 \in \mathfrak{M} \) realising \( t(c/\mathfrak{A}_1 \cup \bar{a}) \) such that

\[
\text{tp}^w(c_0/\mathfrak{A} \cup \bar{a}) \neq \text{tp}^w(c_1/\mathfrak{A} \cup \bar{a}).
\]

By definition of weak type, we can find \( A_{n+1} \) finite, containing \( A_n \cup a_n \) such that (4) holds.

This construction is enough: By Corollary 3.5 and (3), for each \( \eta \in \omega \) there exists \( d_\eta \) realising \( \text{tp}^w(c_\eta/\mathfrak{A}_\eta \cup \bar{a}) \), for \( n < \omega \). We note that finite character is necessary here, since the finite sets \( A_n \) are usually not models. Let \( \eta \neq \nu \in \omega \) and let \( n \) maximal such that \( \eta \upharpoonright n = \nu \upharpoonright n \) but \( \eta(n) \neq \nu(n) \). Then by (4) and definition of \( d_\eta \), we have

\[
\text{tp}^w(d_\eta/A_{n+1}) = \text{tp}^w(c_{\eta+1}/A_{n+1}) \neq \text{tp}^w(c_{\nu+1}/A_{n+1} = \text{tp}^w(d_\nu/A_{n+1}).
\]

This implies that there are continuum many types over \( \mathfrak{A} \) which contradicts \( \mathfrak{N}_0 \)-stability.

**Lemma 3.11.** Assume that \( \mathfrak{A} = \{ \bar{a}_i : i < \omega \} \) is a countable model and \( \bar{a} \) a finite tuple. Then there are \( \bar{b}_i, i < \omega \) and finite \( A_i, i < \omega \), such that:

1. \( \bar{b}_0 = \bar{a} \) and \( A_0 = \emptyset \),
2. \( \bar{a}_n \cup A_n \subseteq A_{n+1} \subseteq \mathfrak{A} \), and \( \bar{b}_n \upharpoonright m = \bar{b}_m \) for all \( m < n < \omega \)
3. \( \text{tp}^w(\bar{b}_n/\mathfrak{A} \cup \bar{a}) \) is weakly isolated over \( A_{n+1} \cup \bar{a} \).
4. \( \mathfrak{A} \cup \bigcup_{i<\omega} \bar{b}_i \) is an \( \mathfrak{N}_0 \)-saturated model.

**Proof.** We construct an increasing sequence of finite subsets \( A_n \subseteq \mathfrak{A} \) and an increasing sequence of finite tuples \( \bar{b}_n \) such that items (1)-(4) hold. We do this by induction on \( n < \omega \). For \( n = 0 \), do as in item (1). Assume that \( \bar{b}_j, A_j \) have been constructed for \( j \leq n \). By \( \mathfrak{N}_0 \)-stability, we can find \( \{ c_i^j : i < \omega, j \leq n \} \) realizing all the Galois types over \( A_j \cup \bar{b}_j \), for \( j \leq n \). Let \( \bar{d}_n = (c_i^j)_{i,j \leq n} \). By Lemma 3.10 there exists \( A_{n+1} \) finite with \( A_n \subseteq A_{n+1} \subseteq \mathfrak{A} \) and there exists \( \bar{b}' \cup \bar{d}' \) realizing \( \text{tp}^w(\bar{b}'/\mathfrak{A}_n \cup \bar{a}) \) such that \( \text{tp}^w(\bar{b}'/\mathfrak{A} \cup \bar{a}) \) is weakly isolated over \( A_{n+1} \cup \bar{a} \). We may assume that \( A_{n+1} \) contains \( a_n \). Since \( \text{tp}^w(\bar{b}'/A_{n+1} \cup \bar{a}) \neq \text{tp}^w(b_n/A_{n+1} \cup \bar{a}) \) by induction hypothesis, we may also assume that \( b' = b_n \). Let \( b_{n+1} = \bar{b}_n \cup \bar{d}' \). Then (1), (2), (3) are satisfied, Finite character is used to show (4) in the form of Lemma 3.8. Let \( c \in \mathfrak{M} \) and \( B \subseteq \mathfrak{A} \cup \bigcup_{n<\omega} \bar{b}_n \) finite. By (2), \( \mathfrak{A} = \bigcup_{n<\omega} A_n \), so there exists \( n < \omega \) such that \( B \subseteq A_n \cup \bar{b}_n \). Then \( \text{tp}^w(c/A_n \cup \bar{b}_n) \) is realized by some \( c_n \), and hence belongs to \( b_{j+1} \) for some \( j \). We are done with the construction.

Finally we prove the result that equality of weak types and equality of Galois types coincide over countable models.
Theorem 3.12. Let \( (\mathcal{K} \leqslant \mathcal{K}) \) be an \( \aleph_0 \)-stable finitary AEC. Assume that \( \mathcal{A} \) is a countable model and \( \text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \). Then also \( \text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A}) \).

Proof. Let \( \mathcal{B} \) be a model as in Lemma 3.11, containing \( \mathcal{A} \cup \bar{a} \). We will find an embedding \( f : \mathcal{B} \rightarrow \mathcal{M} \) with \( f \upharpoonright \mathcal{A} = \text{id}_\mathcal{A} \) such that \( f(a) = b \), and for each finite \( \bar{c} \in \mathcal{B} \),

\[
\text{tp}^w(\bar{c}/\emptyset) = \text{tp}^w(f(\bar{c})/\emptyset).
\]

This is enough by finite character: By \( \mathcal{K} \)-homogeneity there exists an automorphism \( F \) of \( \mathcal{M} \) extending \( f \), so

\[
\text{tp}^g(a/\mathcal{A}) = \text{tp}^g(b/\mathcal{A}).
\]

In order to do this, we construct an increasing sequence of tuples

\[
(\bar{b}_n' : n < \omega),
\]

such that \( b_0' = b \) and for each \( n < \omega \) we have

(*) \[
\text{tp}^w(b_n'/\mathcal{A}) = \text{tp}^w(b_n'/\mathcal{A}).
\]

For \( n = 0 \) let \( b_0' = b \). Then (*) holds by assumption since \( b_0 = a \). Suppose we have constructed \( \bar{b}_n' \) such that (*) holds. We have in particular \( \text{tp}^w(b_n/A_{n+1}) = \text{tp}^w(\bar{b}_n'/A_{n+1}) \) so there is an automorphism \( F \) of \( \mathcal{M} \) which is the identity on \( A_{n+1} \) such that \( F(\bar{b}_n') = \bar{b}_n' \). Let \( b_{n+1}' = F(b_{n+1}) \). We claim that (*) holds for \( b_{n+1}' \). We assume the contrary and let \( B \subset \mathcal{A} \) be finite such that \( \text{tp}^w(b_{n+1}'/B) \neq \text{tp}^w(b_{n+1}/B) \).

We may assume that \( A_{n+1} \subset B \). Let \( g \in \text{Aut}(\mathcal{M}/B) \) be such that \( g(b_{n+1}') = \bar{b}_n \). Now

\[
\text{tp}^w(g(\bar{b}_{n+1}')/A_{n+1}) = \text{tp}^w(b_{n+1}'/A_{n+1}) = \text{tp}^w(b_{n+1}/A_{n+1}),
\]

and thus since \( g(b_0') = \bar{b}_0 = \bar{a} \),

\[
\text{tp}^w(g(\bar{b}_{n+1}')/A_{n+1} \cup \bar{a}) = \text{tp}^w(b_{n+1}/A_{n+1} \cup \bar{a}).
\]

But \( \text{tp}^w(b_{n+1}/\mathcal{A} \cup \bar{a}) \) is weakly isolated over \( A_{n+1} \cup \bar{a} \), and we get that

\[
\text{tp}^w(g(\bar{b}_{n+1}')/B) = \text{tp}^w(b_{n+1}/B).
\]

Since \( \text{tp}^w(g(\bar{b}_{n+1}')/B) = \text{tp}^w(b_{n+1}/B) \), we have a contradiction. This shows the claim.

This construction shows that there exists an isomorphism \( f : \mathcal{B} \rightarrow \mathcal{A} \cup \bigcup_{n<\omega} \bar{b}_n' \) which is the identity on \( \mathcal{A} \), preserving types of finite tuples, such that \( f(a) = b \). \( \square \)

Now we can improve the result of Corollary 3.5. If \( \mathcal{A} \) is a model of size \( \aleph_1 \) and we have a set of coherent types over all finite subsets of \( \mathcal{A} \), we can find \( \bar{a} \) such that \( \text{tp}^w(\bar{a}/\mathcal{A}) \) extends all the types. We note that such type is also unique, due to the finite character of weak types.
Lemma 3.13. Assume $\mathcal{A}$ is a model, $|\mathcal{A}| \leq \aleph_1$ and for each finite $A \subset \mathcal{A}$ there is $\bar{a}_A$ such that if $B \subset A$, then $\text{tp}^w(\bar{a}_B/B) = \text{tp}^w(\bar{a}_A/B)$. Then there is $\bar{a}$ such that for each finite $A \subset \mathcal{A}$, $\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{a}_A/A)$.

Proof. Let $\mathcal{A} = \bigcup_{i<\omega_1} \mathcal{A}_i$, where $(\mathcal{A}_i)_{i<\omega_1}$ is an $\leq_{\mathbb{K}}$-increasing chain of countable models such that $\mathcal{A}_\alpha = \bigcup_{i<\alpha} A_i$, when $\alpha$ is a limit ordinal. From Corollary 3.5 we get for each $i < \omega_1$ a tuple $\bar{a}_i$ such that $\text{tp}^\alpha(\bar{a}_i/A) = \text{tp}^\alpha(\bar{a}_A/A)$ for each finite $A \subset \mathcal{A}_i$. Now if $j < i$, we get from finite character of weak types that $\text{tp}^w(\bar{a}_j/\mathcal{A}_j) = \text{tp}^w(\bar{a}_i/\mathcal{A}_i)$. Then we get from Theorem 3.12 that also $\text{tp}^\beta(\bar{a}_i/\mathcal{A}_j) = \text{tp}^\beta(\bar{a}_j/\mathcal{A}_j)$.

Then we do a similar construction as in Lemma 3.4. We define automorphisms $g_i$, $i < \omega_1$, such that

1. For $j < i < \omega_1$, $g_i \upharpoonright \mathcal{A}_j = g_j \upharpoonright \mathcal{A}_j$ and
2. $g_i(\bar{a}_i) = \bar{a}_0$.

Let $g_0 = \text{id}_M$. Assume we have defined $g_i$ for $i < \alpha$.

Case 1: $\alpha = \beta + 1$. Since $\text{tp}^\beta(\bar{a}_\alpha/\mathcal{A}_\beta) = \text{tp}^\beta(\bar{a}_\beta/\mathcal{A}_\beta)$, also $\text{tp}^\beta(g_\beta(\bar{a}_\alpha)/g_\beta(\mathcal{A}_\beta)) = \text{tp}^\beta(g_\beta(\bar{a}_\beta)/g_\beta(\mathcal{A}_\beta))$, and we have an automorphism $f$ such that $f \upharpoonright g_\beta(\mathcal{A}_\beta)$ is the identity and $f(g_\beta(\bar{a}_\alpha)) = g_\beta(\bar{a}_\beta) = \bar{a}_0$. We can take $g_\alpha = f \circ g_\beta$.

Case 2: $\alpha$ is a limit ordinal. The mapping $\bigcup_{i<\alpha}(g_i \upharpoonright \mathcal{A}_i) : \mathcal{A}_\alpha \to M$ extends to an automorphism $F$. Every finite $A \subset \mathcal{A}_\alpha$ is included in some $\mathcal{A}_i$ for $i < \alpha$, and $g_i^{-1} \circ F$ shows that $\text{tp}^\alpha(F^{-1}(\bar{a}_0)/A) = \text{tp}^\alpha(\bar{a}_\alpha/A)$. Thus $\text{tp}^w(F^{-1}(\bar{a}_0)/\mathcal{A}_\alpha) = \text{tp}^w(\bar{a}_\alpha/\mathcal{A}_\alpha)$ and by Theorem 3.12, $\text{tp}^\beta(F^{-1}(\bar{a}_0)/\mathcal{A}_\alpha) = \text{tp}^\beta(\bar{a}_\alpha/\mathcal{A}_\alpha)$. Let $f \in \text{Aut}(M/\mathcal{A}_\alpha)$ be such that $f(\bar{a}_\alpha) = F^{-1}(\bar{a}_0)$. We can take $g_\alpha = F \circ f$.

Finally the mapping $\bigcup_{i<\omega_1}(g_i \upharpoonright \mathcal{A}_i) : \mathcal{A} \to M$ extends to an automorphism $G$. We can take $\bar{a} = G^{-1}(\bar{a}_0)$. Then for each $i < \omega_1$, automorphism $G^{-1} \circ g_i$ shows that $\text{tp}^\beta(\bar{a}_i/\mathcal{A}_i) = \text{tp}^\beta(\bar{a}/\mathcal{A})$. Thus when $A \subset \mathcal{A}$ finite, there is some $i < \alpha$ such that $A \subset \mathcal{A}_i$. Then $\text{tp}^\beta(\bar{a}_A/A) = \text{tp}^\beta(\bar{a}_1/A) = \text{tp}^\beta(\bar{a}/A)$.

3.1. Splitting and $\aleph_0$-saturation. In the following we define our notion of splitting for weak types. This differs from the notion of splitting for Galois types over models, which has been traditionally studied in the context of abstract elementary classes, and thus we might call this notion weak splitting. Although, to shorten the notation, we choose to call it splitting and just say that a weak type splits over a finite set.
Definition 3.14 (Splitting). We say that the weak type $\text{tp}^w(\bar{a}/A)$ splits over finite $B \subset A$ if there are $\bar{c}, \bar{d} \in A$ such that
\[
\text{tp}^w(\bar{c}/B) = \text{tp}^w(\bar{d}/B) \quad \text{but} \quad \text{tp}^w(\bar{c}/B \cup \{\bar{a}\}) \neq \text{tp}^w(\bar{d}/B \cup \{\bar{a}\}).
\]
We say that such $\bar{c}, \bar{d}$ witness the fact.

We now define a first notion of independence based on splitting.

Definition 3.15 (Independence). We write that $\bar{a} \downarrow^s A B$ if $\text{tp}^w(\bar{a}/A \cup B)$ does not split over a finite subset of $A$.

If $\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{b}/A)$ and $B \subset A$ is finite, then $\text{tp}^w(\bar{a}/A)$ splits over $B$ if and only if $\text{tp}^w(\bar{b}/A)$ splits over $B$. Also if $\text{tp}^w(\bar{a}/A)$ splits over finite $E \subset A$ and $E' \subset E$, then $\text{tp}^w(\bar{a}/A)$ splits over $E'$.

The proof for the following theorem is standard for $\aleph_0$-Galois-stable AEC with amalgamation and joint embedding. Only here we have an a priori weaker notion of $\aleph_0$-stability, and thus use the stronger version of the lemma considering unions of Galois types, provided by finite character. However, finite character is essential in proving this property for weak types.

Theorem 3.16. Assume that $(\mathfrak{K}, \preceq_\mathfrak{K})$ is an $\aleph_0$-stable finitary AEC. Let $\mathcal{A}$ be a model. Then $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over a finite subset of $\mathcal{A}$.

Proof. We first observe, that whenever $\text{tp}^w(\bar{a}/A)$ splits over finite $E \subset A$, there is $f \in \text{Aut}(\mathfrak{M}/E)$ and finite $E' \subset A \cap F(A)$ containing $E$ such that
\[
\text{tp}^w(\bar{a}/E') \neq \text{tp}^w(f(\bar{a})/E')
\]
and both extend $\text{tp}^w(\bar{a}/E)$. This is true, since if $\bar{c}, \bar{d} \in A$ witness the splitting, we can take $f \in \text{Aut}(\mathfrak{M}/E)$ mapping $\bar{c}$ to $\bar{d}$, and then take $E' = E \cap \{\bar{d}\} \subset A \cap f(A)$. If $\bar{a}$ and $f(\bar{a})$ would have same weak type over $E'$, we would gain $g \in \text{Aut}(\mathfrak{M}/E \cap \{\bar{a}\})$ mapping $\bar{c}$ to $\bar{d}$, a contradiction.

Now suppose that $\text{tp}^w(c/\mathcal{A})$ splits over every finite subset of $\mathcal{A}$. We first show that we may assume, without loss of generality, that $\mathcal{A}$ is countable: Construct an $\preceq_\mathfrak{K}$-increasing sequence $(\mathcal{A}_n : n < \omega)$ of countable submodels of $\mathcal{A}$ such that $\text{tp}^w(c/\mathcal{A}_{n+1})$ splits over every finite subset of $\mathcal{A}_n$: This is possible since having constructed $\mathcal{A}_n$, the weak type $\text{tp}^w(c/\mathcal{A}_n)$ splits over every finite subset of $\mathcal{A}_n$. Since there are only countably many finite subsets of $\mathcal{A}_n$, we can choose countably many witnesses to splitting and find $\mathcal{A}_{n+1}$ extending $\mathcal{A}_n$ containing all these witnesses.
Then $\text{tp}^w(a/\mathcal{A}_{n+1})$ is as desired. This is enough: Let $\mathcal{A}' = \bigcup_{n<\omega} \mathcal{A}_n$. Then $\mathcal{A}'$ is countable and $\text{tp}^w(a/\mathcal{A}')$ splits over every finite subset of $\mathcal{A}'$.

So assume that $\mathcal{A}$ is countable and write $\mathcal{A} = \{a_n : n < \omega\}$. We construct finite partial isomorphisms $f_\eta : A_\eta \to B_\eta$, and automorphisms $F_\eta$ extending $f_\eta$, for $\eta \in <\omega 2$, such that letting $c_\eta = F_\eta(c)$ we have:

1. $A_\emptyset = B_\emptyset = \{a_0\}$, and $f_\emptyset \subseteq F_\emptyset = \text{id}$.
2. $a_n \in A_\eta$ if $n = \ell(\eta)$ and $A_\eta \subseteq \mathcal{A}$ is finite.
3. If $\eta < \nu$ then $f_\eta \subseteq f_\nu$.
4. If $\eta < \nu$ then $c_\nu$ realizes $\text{tp}^w(c_\eta/B_\eta)$.
5. $B_{\eta^0} = B_{\eta^{-1}}$ and $\text{tp}^w(c_{\eta^0}/B_{\eta^0}) \neq \text{tp}^w(c_{\eta^{-1}}/B_{\eta^{-1}})$.
6. If $\eta \in \omega 2$ then $\bigcup_{n<\omega} B_{\eta|n}$ is a model.

This is enough: For each $\eta \in \omega 2$ there exists $d_\eta$ realizing $\text{tp}^w(c_{\eta|n}/B_{\eta|n})$, for each $n < \omega$ by (4), (6) and Corollary 3.5. Let $B = \bigcup_{\eta \in \omega 2} B_\eta$. Then $B$ is a countable set and $\text{tp}^w(d_\eta/B) \neq \text{tp}^w(d_\nu/B)$ for $\eta \neq \nu$ by (5), contradicting $\aleph_0$-stability.

This is possible: Suppose that $f_\eta : A_\eta \to B_\eta$ and $F_\eta$ have been constructed. Since $\text{tp}^w(c/\mathcal{A})$ splits over every finite set, then $\text{tp}^w(c_\eta/F_\eta(\mathcal{A}))$ splits over $B_\eta$. Hence, by the above observation, there is $B' \subseteq F_\eta(\mathcal{A})$ finite containing $B_\eta$ and $G$ an automorphism which is the identity on $B_\eta$ such that

$$\text{tp}^w(a_\eta/B') \neq \text{tp}^w(G(a_\eta)/B').$$

Let $B_{\eta^{-1}} = B_{\eta^{-1}} = B'$, and let $A_{\eta^{-1}} = A_{\eta^{-1}} = F_{\eta^{-1}}(B')$, which we may assume contains $a_{\eta^{-1}}$ by monotonicity. Let $F_{\eta^{-1}} = F_{\eta}$, $F_{\eta^{-1}} = G \circ F_{\eta}$, and define $f_\eta$ and $f_{\eta^{-1}}$ by restriction (both extend $f_\eta$ since $G$ is the identity on $B_\eta$). To see (6), let $\eta \in \omega 2$. The mapping

$$f_\eta = \bigcup_{n<\omega} f_{\eta|n} : \mathcal{A} \to \bigcup_{n<\omega} B_{\eta|n}$$

preserves types of finite tuples, and thus is a $\mathbb{K}$-embedding by finite character.

As a corollary of Theorem 3.16, we get that there is no $\bar{a}$ and $(A_n : n < \omega)$ such that $\bigcup_{n<\omega} A_n$ is a model and $\text{tp}^w(\bar{a}/A_{n+1})$ splits over every finite $E \subseteq A_n$.

In the following theorem we prove some basic properties for splitting.

**Theorem 3.17.** Let $(\mathbb{K}, \leq_{\mathbb{K}})$ be a finitary AEC, stable in $\aleph_0$.

1. **Monotonicity** If $A \subseteq B \subseteq C \subseteq D$, then $\bar{a} \downarrow_A C$.
2. **Invariance** If $f$ is an automorphism of $\mathfrak{M}$, $\bar{a} \downarrow_A B$ if and only if $f(\bar{a}) \downarrow_{f(A)} f(B)$.
3. **Local character** For each model $\mathcal{A}$ and a finite sequence $\bar{a}$ there is finite $E \subseteq \mathcal{A}$ such that $\bar{a} \downarrow_E \mathcal{A}$. 


(4) **Countable extension** Let $\mathcal{A}$ be a countable $\aleph_0$-saturated model. Let $B$ be countable containing $\mathcal{A}$. For each $\bar{a}$ there is $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{A})$ such that $\bar{b} \downarrow^w_{\mathcal{A}} B$. Moreover, if $tp^w(\bar{a}/\mathcal{A})$ does not split over the finite subset $E$, then $tp^w(\bar{b}/B)$ does not split over the finite set $E$.

(5) **Stationarity** Assume $\mathcal{A}$ is an $\aleph_0$-saturated model and $A \subset B$. If $tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow^w_{\mathcal{A}} B$ and $\bar{b} \downarrow^w_{\mathcal{A}} B$, then $tp^w(\bar{a}/B) = tp^w(\bar{b}/B)$.

(6) **Transitivity** Let $A \subset B \subset C$ and $\mathcal{B}$ be an $\aleph_0$-saturated model. Then $\bar{a} \downarrow^w_A C$ if and only if $\bar{a} \downarrow^w_A B$ and $\bar{a} \downarrow^w_A C$.

(7) **Finite character** Let $E$ be finite and $E \subset B$. Then $\bar{a} \downarrow^w_E B$ if and only if $\bar{a} \downarrow^w_E B_0$ for every finite $B_0 \subset B$.

The same holds if instead of $E$ we have an $\aleph_0$-saturated model $\mathcal{A}$.

**Proof.** Monotonicity and Invariance are clear from the definition of independence. Local character is Theorem 3.16.

**Countable extension:** By definition and Theorem 3.16, it is enough to prove the second statement.

By monotonicity, we may assume that $B = \mathcal{B}$ is an $\aleph_0$-saturated model. Since both $\mathcal{A}$ and $\mathcal{B}$ are countable and $\aleph_0$-saturated, there is $f \in \text{Aut}(\mathcal{M}/E)$ such that $f(\mathcal{A}) = \mathcal{B}$. Now $tp^w(f(\bar{a})/\mathcal{B})$ does not split over $E$ by invariance.

Let $C \subset \mathcal{A}$ be an arbitrary finite subset. Then $C \cup f(C) \subset \mathcal{B}$. Since $tp^w(f(\bar{b})/\mathcal{B})$ does not split over $E$, there is $h \in \text{Aut}(\mathcal{M}/E \cup \{f(\bar{b})\})$ such that $h \upharpoonright C = f \upharpoonright C$. The automorphism $f^{-1} \circ h$ maps $f(\bar{a})$ to $\bar{a}$ and fixes $C$ pointwise. Since $C$ was arbitrary, we get that $tp^w(f(\bar{a})/\mathcal{A}) = tp^w(\bar{a}/\mathcal{A})$, and may take $\bar{b} = f(\bar{a})$.

**Stationarity:** Let $C \subset B$ be an arbitrary finite set. Let $E_\bar{a} \subset \mathcal{A}$ be a finite set such that $tp^w(\bar{a}/B)$ does not split over $E_\bar{a}$ and similarly $E_\bar{b} \subset \mathcal{A}$ for $tp^w(\bar{b}/B)$. Since $\mathcal{A}$ is $\aleph_0$-saturated, we have $f \in \text{Aut}(\mathcal{M}/E_\bar{a} \cup E_\bar{b})$ mapping $f(C)$ into $\mathcal{A}$. Then by the choice of $E_\bar{a}$ we have an automorphism $f_\bar{a} \in \text{Aut}(\mathcal{M}/E_\bar{a} \cup \{\bar{a}\})$ such that $f_\bar{a} \upharpoonright C = f \upharpoonright C$. Similarly we get $f_\bar{b} \in \text{Aut}(\mathcal{M}/E_\bar{b} \cup \{\bar{b}\})$ such that $f_\bar{b} \upharpoonright C = f \upharpoonright C$. Finally we use the assumption that $tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A})$ to get an automorphism $g \in \text{Aut}(\mathcal{M}/f(C))$ sending $\bar{a}$ to $\bar{b}$. When we combine these mappings we get an automorphism $h = f_\bar{b}^{-1} \circ g \circ f_\bar{a} \in \text{Aut}(\mathcal{M}/C)$ such that $h(\bar{a}) = \bar{b}$. Thus $tp^w(\bar{a}/C) = tp^w(\bar{b}/C)$ and since $C \subset B$ was an arbitrary finite set, we get that $tp^w(\bar{a}/B) = tp^w(\bar{b}/B)$.

**Transitivity:** The ”$\Rightarrow$”-direction follows from monotonicity.

We prove the other direction first assuming that all the sets are countable. Let $E \subset A$ be a finite set such that $tp^w(\bar{a}/\mathcal{B})$ does not split over $E$. We use countable extension to get $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{B})$ such that $tp^w(\bar{b}/C)$ does not split over $E$. 


Since also $\bar{a} \downarrow_{\mathcal{B}} C$, we get from stationarity that $tp^w(\bar{b}/C) = tp^w(\bar{a}/C)$. Hence $tp^w(\bar{a}/C)$ does not split over $E \subset A$.

Now we prove the same for $A, \mathcal{B}$ and $C$ of arbitrary size. By definition, there exists finite $E \subset A$ such that $\bar{a} \downarrow_{\mathcal{B}^0} E$. Again by definition and monotonicity there exists an $\aleph_0$-saturated countable $\mathcal{B}' \subset \mathcal{B}$ such that $\bar{a} \downarrow_{\mathcal{B}'} E$. By monotonicity, it is enough to show that $\bar{a} \downarrow_{E} C$. If this fails, by definition there is finite $C_0 \subset C$ such that $\bar{a} \downarrow_E C_0$. Again by monotonicity, we have $\bar{a} \downarrow_{E} C'$ and $\bar{a} \downarrow_{\mathcal{B}'} C' \cup C_0$.

But now $E \subset \mathcal{B}' \subset \mathcal{B}' \cup C_0$ are countable, and we get by the countable case and monotonicity that $\bar{a} \downarrow_E C_0$, which contradicts the choice of $C_0$.

**Finite character:** The first statement follows immediately from the definition and monotonicity. We prove the second statement.

Other direction is clear by monotonicity. Assume that $\bar{a} \downarrow_{\mathcal{B}} B_0$ for every finite $B_0 \subset B$. By Theorem 3.16 there is finite $E \subset A$ such that $\bar{a} \downarrow_{E} A'$. Let $B_0 \subset B$ be finite. Then $E \subset A' \subset A \cup B_0$, $\bar{a} \downarrow_{E} A'$ and $\bar{a} \downarrow_{\mathcal{B}} A' \cup B_0$. We get by transitivity that $tp^w(\bar{a}/A \cup B_0)$ does not split over $E$. Since this holds for every finite $B_0 \subset B$, we get that $t^w(\bar{a}/A \cup B)$ does not split over $E$.

We can prove a stronger version of Theorem 3.17(4). The proof is similar.

**Lemma 3.18.** Assume $\mathcal{B} \subset C$ are countable and $\mathcal{B}$ is a $\aleph_0$-saturated model. Let $A = (a_i)_{i<\omega}$ be a set. There is $A' = (a'_i)_{i<\omega}$ such that for each $n < \omega$ $tp^w((a_0, ..., a_n)/\mathcal{B}) = tp^w((a'_0, ..., a'_n)/\mathcal{B})$ and $(a'_0, ..., a'_n) \downarrow_{\mathcal{B}} C$.

This we denote $tp^w(A/\mathcal{B}) = tp^w(A'/\mathcal{B})$ and $A' \downarrow_{\mathcal{B}} C$.

Since Galois types and weak types agree over countable models, we can actually improve Theorem 3.17(4) to include extensions of types over $\aleph_0$-saturated models up to size $\aleph_1$.

**Remark 3.19.** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. Assume that $tp^w(\bar{a}/\mathcal{A})$ does not split over a finite subset $E$. Let $B$ be a set containing $\mathcal{A}$ of size $\aleph_1$. Then there exists $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{A})$ such that $tp^w(\bar{b}/B)$ does not split over $E$.

**Proof.** By monotonicity, we may assume that $B = \mathcal{B}$ is an $\aleph_0$-saturated model.

Let $\mathcal{A}_0 \preccurlyeq_{\mathcal{K}} \mathcal{A}$ be countable such that $E \subset \mathcal{A}_0$. For every finite $B \subset \mathcal{B}$ we get from countable extension some $\bar{b}_B$ such that $tp^w(\bar{b}_B/\mathcal{A}_0) = tp^w(\bar{a}/\mathcal{A}_0)$ and $tp^w(\bar{b}_B/\mathcal{A}_0 \cup B)$ does not split over $E$. When $B$ and $B'$ are finite and $B \subset B' \subset \mathcal{B}$, we have that $tp^w(\bar{b}_B/\mathcal{A}_0) = tp^w(\bar{b}_{B'}/\mathcal{A}_0)$, $\bar{b}_B \downarrow_{\mathcal{A}_0} B$ and $\bar{b}_{B'} \downarrow_{\mathcal{A}_0} B$. Thus $tp^w(\bar{b}_B/\mathcal{A}_0 \cup B) = tp^w(\bar{b}_{B'}/\mathcal{A}_0 \cup B)$ by stationarity. Hence we may use Lemma 3.13 to get such $\bar{b}$ that $tp^w(\bar{b}/B) = tp^w(\bar{b}_B/B)$ for every finite $B \subset \mathcal{B}$. By finite character of splitting, $tp^w(\bar{b}/\mathcal{B})$ does not split over $E$. 


Then we see that actually $\text{tp}^w(\bar{b}/\mathcal{A}) = \text{tp}^w(\bar{a}/\mathcal{A})$. When $\mathcal{A} \subset \mathcal{A}$ is a finite subset, we have that $\text{tp}^w(\bar{a}/\mathcal{A}_0 \cup A)$ does not split over $E$. Then again from stationarity we get that $\text{tp}^w(\bar{a}/\mathcal{A}_0 \cup A) = \text{tp}^w(\bar{b}_A/\mathcal{A}_0 \cup A)$ and thus $\text{tp}^g(\bar{a}/A) = \text{tp}^g(\bar{b}_A/A) = \text{tp}^g(\bar{b}/A)$. □

4. Extension property

In this section $(\mathbb{K}, \preccurlyeq)$ is $\aleph_0$-stable finitary AEC. With only the assumptions we have set up so far, we can not prove that our notion of independence has all the properties of an independence calculus even over $\aleph_0$-saturated models. Especially we need symmetry over an $\aleph_0$-saturated model. This we can gain if we assume $\lambda$-extension property, formulated in Definition 4.1, for $\lambda = \mathbb{H}$. The number $\mathbb{H}$ is the Hanf number for abstract elementary classes with a countable Löwenheim-Skolem number, see Lemma 2.22. In section 4.1 we prove symmetry using this property as an assumption and then we show that $\mathbb{H}$-extension implies $\lambda$-extension for arbitrary $\lambda$. In sections 4.3 and 4.4 we discuss what more natural assumptions would imply this property. We also consider the existence of weakly saturated models.

**Definition 4.1 ($\lambda$-extension property).** We say that $(\mathbb{K}, \preccurlyeq)$ has the $\lambda$-extension property if the following holds:

Let $\mathcal{A}$ be an $\aleph_0$-saturated model and let $B$ contain $\mathcal{A}$, $|B| < \lambda$. Assume that $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over finite subset $E$. Then there exists $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\text{tp}^w(\bar{b}/B)$ does not split over $E$.

We note that when $\mathcal{A}$ is a model, by Theorem 3.16, there always exists a finite subset $E$ such that $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over $E$. We say that $(\mathbb{K}, \preccurlyeq)$ has the extension property, if it has the $\mathbb{H}$-extension property. Then it also has $\lambda$-extension property for every $\lambda$, by Proposition 4.4.

4.1. **Symmetry.** The proof for symmetry is standard from first order and homogeneous model theory. We make a counter assumption, define a long linear ordering using $\mathbb{H}$-extension property and finally contradict $\aleph_0$-stability. Here we use the homogeneity of the model $\mathfrak{M}^*$.

After proving symmetry we use it to show that $\mathbb{H}$-extension property implies $\lambda$-extension for all $\lambda$.

**Lemma 4.2.** Assume that $(\mathbb{K}, \preccurlyeq)$ satisfies the $\lambda$-extension property. Assume that $\mathcal{A}$ is a countable $\aleph_0$-saturated model, $\bar{a} \preccurlyeq_{\mathcal{A}} \bar{b}$ and $\bar{b} \not\preccurlyeq_{\mathcal{A}} \bar{a}$. Then there exists a sequence $(\bar{a}_i, \bar{b}_i)_{i<\lambda}$ of length $\lambda$ such that $\bar{b}_i \preceq_{\mathcal{A}} \bar{a}_j$ if and only if $i > j$. 
Proof. We construct such a sequence by induction. Let \( a_0 = a \) and \( b_0 = b \). Assume we have found \( a_i, b_i \) for each \( i < \alpha \). Now we use Theorem 3.16 and the \( \lambda \)-extension property to get \( a_\alpha \) and \( b_\alpha \) such that

(1) \( \text{tp}^w(a_\alpha \sim b_\alpha / \mathcal{A}) = \text{tp}^w(a_0 \sim b_0 / \mathcal{A}) \)

(2) \( a_\alpha \sim b_\alpha \Vdash \bigcup_{i<\alpha} \{a_i, b_i\} \).

From monotonicity we get that \( a_\alpha \sim b_\alpha \Vdash a_i \) for each \( i < \alpha \) and thus \( b_\alpha \Vdash a_i \) for each \( i < \alpha \). First we claim that

(3) when \( \beta \leq \alpha \), \( \text{tp}^w(a_\alpha \sim b_\beta / \mathcal{A}) = \text{tp}^w(a_0 \sim b_0 / \mathcal{A}) \).

The proof of this claim is much similar to the proof of stationarity in 3.17. If \( \beta = \alpha \), the claim follows from the definition of \( a_\alpha \) and \( b_\alpha \). Thus let \( \beta < \alpha \). Since \( a_0 \sim b_0 \) and \( a_\alpha \Vdash b_\beta \), we have finite \( E_1, E_2 \subset \mathcal{A} \) such that

(a) \( \text{tp}^w(a_0 / \mathcal{A} \cup \{b_0\}) \) does not split over \( E_1 \)

(b) \( \text{tp}^w(a_\alpha / \mathcal{A} \cup \{b_\beta\}) \) does not split over \( E_2 \).

Let \( C \subset \mathcal{A} \) be an arbitrary finite set. Since \( \mathcal{A} \) is \( \aleph_0 \)-saturated, there exists an automorphism \( f \in \text{Aut}(\mathcal{M}/E_1E_2C) \) such that \( f(b_0) \subset \mathcal{A} \). From (a) we get an automorphism such that \( h_1 \upharpoonright C \cup \{b_0\} = f \upharpoonright C \cup \{b_0\} \) and \( h_1(a_0) = a_0 \).

Now we use the fact that \( \text{tp}^w(b_\beta / \mathcal{A}) = \text{tp}^w(b_0 / \mathcal{A}) \) to get an automorphism \( f' \in \text{Aut}(\mathcal{M}/C \cup E_2) \) such that \( f'(b_\beta) = b_0 \). Then \( (f \circ f')(b_\beta \sim C) = f(b_0 \sim C) \). Thus from (b) we get an automorphism \( h_2 \in \text{Aut}(\mathcal{M}/C \cup \{a_\alpha\}) \) such that \( h_2(b_\beta) = f(b_0) \).

Since \( \text{tp}^w(a_0 / \mathcal{A}) = \text{tp}^w(a_\alpha / \mathcal{A}) \), there is also \( h \in \text{Aut}(\mathcal{M}/\{f(b_0)\} \cup C) \) such that \( h(a_0) = a_\alpha \).

Finally we combine these automorphisms to \( h_2^{-1} \circ h \circ h_1 \in \text{Aut}(\mathcal{M}/C) \). Now \( (h_2^{-1} \circ h \circ h_1)(a_0, b_0) = h_2^{-1}(h(a_0, f(b_0))) = h_2^{-1}(a_\alpha, f(b_0)) = (a_\alpha, b_\beta) \). Since \( C \subset \mathcal{A} \) was arbitrary, this proves claim (3).

Now we want to show that

(4) for all \( i \leq \alpha \), \( b_i \not\equiv_{\mathcal{A}} a_\alpha \).

To prove this, we assume the contrary. Let \( \beta \leq \alpha \) and \( E \subset \mathcal{A} \) be a finite set such that \( \text{tp}^w(b_\beta / \mathcal{A} \cup \{a_\alpha\}) \) does not split over \( E \). We have that \( b_0 \not\equiv_{\mathcal{A}} a_0 \) and thus \( \text{tp}^w(b_0 / \mathcal{A} \cup \{a_\alpha\}) \) splits over \( E \). Let \( c, d \subset \mathcal{A} \cup \{a_\alpha\} \) witness that.

From (3) we get \( g \in \text{Aut}(\mathcal{M}/(\{c, d\} \cap \mathcal{A}) \cup E) \) such that \( g(a_0, b_0) = (a_\alpha, b_\beta) \).

Since \( g(c) \) and \( g(d) \) are in \( \mathcal{A} \cup \{a_\alpha\} \) and \( \text{tp}^\mathcal{A}(g(c)/E) = \text{tp}^\mathcal{A}(g(d)/E) \) from the choice of \( E \) we get \( g^* \in \text{Aut}(\mathcal{M}/E \cup \{b_\beta\}) \) such that \( g^*(c) = d \).

Now \( (g^{-1} \circ g^* \circ g)(c) = d \) and \( (g^{-1} \circ g^* \circ g) \) is in \( \text{Aut}(\mathcal{M}/E \cup \{b_\beta\}) \), which contradicts the choice of \( c \) and \( d \). This proves (4). \( \square \)
Finally we can prove symmetry. This is the first time we use properties of the homogeneous model $\mathfrak{M}^*$.

**Theorem 4.3 (Symmetry).** Assume that $(\mathbb{K}, \succeq_{\mathbb{K}})$ is $\aleph_0$-stable finitary AEC with $\mathbb{H}$-extension property. Let $\mathcal{A}$ be an $\aleph_0$-saturated model. If $\bar{a} \downarrow^*_{\mathcal{A}} \bar{b}$, then $\bar{b} \downarrow^*_{\mathcal{A}} \bar{a}$.

**Proof.** We assume the contrary. Let $\bar{a}$ and $\bar{b}$ be such that $\bar{a} \downarrow^*_{\mathcal{A}} \bar{b}$ and $\bar{b} \downarrow^*_{\mathcal{A}} \bar{a}$.

By monotonicity and Theorem 3.16 there is countable and $\aleph_0$-saturated $\mathcal{A}_0 \subseteq_{\mathbb{K}} \mathcal{A}$ such that $\bar{a} \downarrow^*_{\mathcal{A}_0} \mathcal{A} \cup \bar{b}$. Then by transitivity and monotonicity again, $\bar{a} \downarrow^*_{\mathcal{A}_0} \bar{b}$ and $\bar{b} \downarrow^*_{\mathcal{A}_0} \bar{a}$. Thus we may assume that $\mathcal{A}$ is countable. Then we get from Lemma 4.2 a sequence $(\bar{a}_i, \bar{b}_i)_{i < \mathbb{H}}$ such that

$$\bar{b}_i \downarrow^*_{\mathcal{A}} \bar{a}_j$$

if and only if $i > j$.

Furthermore we use Lemmas 2.22 and 2.23 to get a sequence $(\bar{a}_i, \bar{b}_i)_{i \in (\mathbb{R}, <)}$ such that

$$\bar{b}_i \downarrow^*_{\mathcal{A}} \bar{a}_j$$

if and only if $j < i$.

Here we use the homogeneity of $\mathfrak{M}^*$ to see that similarity of $\tau^*$-types imply similarity of Galois types, so that the new sequence still has the splitting condition. When we denote $B = \mathcal{A} \cup \{ (\bar{a}_i, \bar{b}_i) : i \in \mathbb{Q} \}$, $B$ is countable and if $i, j \in \mathbb{R}$ and $i \neq j$, tuples $(\bar{a}_i, \bar{b}_i)$ and $(\bar{a}_j, \bar{b}_j)$ have different weak type over $B$. This contradicts the $\aleph_0$-stability assumption. \videopage{26}

We use symmetry and again the model $\mathfrak{M}^*$ to show that $\mathbb{H}$-extension property implies $\lambda$-extension property for every $\lambda$.

**Proposition 4.4.** Assume that $(\mathbb{K}, \succeq_{\mathbb{K}})$ has the $\mathbb{H}$-extension property. Assume that $\mathcal{A}$ is $\aleph_0$-saturated and $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over finite subset $E$. Let $\mathcal{A} \subset B$.

Then there is $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\text{tp}^w(\bar{b}/B)$ does not split over $E$.

**Proof.** Let $\mathcal{A}_0 \subseteq_{\mathbb{K}} \mathcal{A}$ be countable and $\aleph_0$-saturated such that $E \subset \mathcal{A}_0$. For every $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A}_0)$ such that $\bar{b} \downarrow^*_{\mathcal{A}_0} B$ we get by stationarity that $\bar{b}$ also realizes $\text{tp}^w(\bar{a}/\mathcal{A})$. Thus we may assume that $\mathcal{A}$ is countable. Denote $|B|^+ = \kappa$.

Using $\mathbb{H}$-extension property, we define $\bar{a}_i$ and $\mathcal{A}_i$, $i < \mathbb{H}$ such that

1. $\bar{a}_i$ realizes $\text{tp}^w(\bar{a}/\mathcal{A})$ for all $i < \mathbb{H}$,
2. $\mathcal{A}_i$ is the closure of $\mathcal{A} \cup \bigcup_{j < i} \bar{a}_j$ under the functions of $\mathfrak{M}^*$ and
3. $\bar{a}_i \downarrow^*_{\mathcal{A}} \mathcal{A}_i$.

Using Lemmas 2.22 and 2.23 and the homogeneity of $\mathfrak{M}^*$, we can find a sequence $\bar{b}_j$, $j < \kappa$, such that it is $\tau^*$-order-indiscernible over $\mathcal{A}$ and that

$$f \in \text{Aut}(\mathfrak{M}^*/\mathcal{A})$$

such that $f(\bar{b}_k) = \bar{a}_{i_k}$ for each $k \leq n$.

Finally we can prove symmetry. This is the first time we use properties of the homogeneous model $\mathfrak{M}^*$.
First we claim the following:

(3) For each \( \bar{c} \in A \) there is finite \( X \subset \kappa \) such that \( \bar{c} \downarrow^s_{\mathcal{A}} \bar{b}_i \) for each \( i \in \kappa \setminus X \).

If not, then there are \( \bar{c} \in A \) and \( i_0 < \ldots < i_k < \ldots < \kappa \), \( k < \omega \), such that for all \( k < \omega \), \( \bar{c} \downarrow^s_{\mathcal{A}} \bar{b}_{i_k} \). Let \( \mathcal{B}_k \) be the closure of \( \mathcal{A} \cup \bigcup_{j < k} \bar{b}_{i_j} \) under the functions of \( \mathcal{M}^* \). By (4), \( \bar{b}_{i_k} \downarrow^s_{\mathcal{A}} \mathcal{B}_k \). By symmetry, \( \bar{b}_{i_k} \downarrow^s_{\mathcal{A}} \bar{c} \), and by transitivity and symmetry again, \( \bar{c} \downarrow^s_{\mathcal{B}_{i_k}} \bar{b}_{i_k} \) for all \( k < \omega \). We get an increasing chain of models \( \mathcal{B}_k \) such that \( \bar{c} \downarrow^s_{\mathcal{B}_{i_k}} \mathcal{B}_{k+1} \) for all \( k < \omega \), a contradiction. This proves (3).

Since \( \kappa > |A| \), there is \( i < \kappa \) such that \( \bar{c} \downarrow^s_{\mathcal{A}} \bar{b}_i \) for all \( \bar{c} \in A \). But now we can take \( \bar{b} = \bar{b}_i \) by symmetry and finite character of splitting.

The property of (3) can be shown to hold for any Morley sequence \((\bar{b}_i)_{i<\kappa}\) over a countable \( \aleph_0 \)-saturated model, see Definition 5.1.

From now on we will refer to \( \mathbb{H} \)-extension property as extension property.

4.2. Saturated models. We prove some results considering the existence of saturated models. We say that a model \( \mathcal{A} \) is weakly saturated, it every weak type over a subset of size strictly less than \( |\mathcal{A}| \) is realized in \( \mathcal{A} \).

We say that a model \( \mathcal{A} \) is \( \aleph_1 \)-saturated, if for every countable \( A \subset \mathcal{A} \) and \( \bar{b} \), the Galois type \( tp^\mathcal{A}(\bar{b}/A) \) is realized in \( \mathcal{A} \). Due to Theorem 3.12, this is equivalent to saying that every weak type \( tp^\mathcal{A}(\bar{b}/A) \) over countable \( A \subset \mathcal{A} \) is realized in \( \mathcal{A} \).

When \((\mathbb{K}, \leq_{\mathbb{K}})\) is finitary, \( \aleph_0 \)-stability implies \( \kappa \)-stability also for every uncountable cardinal \( \kappa \). The result follows from stationarity as in the first order case.

**Definition 4.5.** We say that \((\mathbb{K}, \leq_{\mathbb{K}})\) is \( \kappa \)-stable, if for every \( |A| \leq \kappa \) and sequence \((\bar{a}_i)_{i<\kappa^+}\), there are \( i_0, j_0 \leq \kappa^+ \) such that \( tp^\mathcal{A}(\bar{a}_{i_0}/A) = tp^\mathcal{A}(\bar{a}_{j_0}/A) \).

**Theorem 4.6.** Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is finitary and stable in \( \aleph_0 \). Then it is also \( \kappa \)-stable for every infinite \( \kappa \).

**Proof.** Let \( A \) be of size \( \kappa \). We may assume that \( A = \mathcal{A} \) is an \( \aleph_0 \)-saturated model. Let \((\bar{a}_i)_{i<\kappa^+}\) be a sequence of tuples. For any \( \bar{a}_i \) there is finite \( E_i \subset \mathcal{A} \) such that \( t^\mathcal{A}(\bar{a}_i/\mathcal{A}) \) does not split over \( E_i \). There are only \( \kappa \)-many finite sets \( E \subset \mathcal{A} \). Then there is a subsequence \((\bar{a}_{i_j})_{j<\kappa^+}\) such that \( t^\mathcal{A}(\bar{a}_{i_j}/\mathcal{A}) \) does not split over the same finite set \( E \) for each \( j < \kappa^+ \). Let \( \mathcal{A}_0 \leq_{\mathbb{K}} \mathcal{A} \) be \( \omega \)-saturated and countable such that \( E \subset \mathcal{A}_0 \). Then by \( \omega \)-stability, there are only countably many weak types over \( \mathcal{A}_0 \). Thus there are some tuples \( \bar{a}_{i_0}, \bar{a}_{i_3}, \alpha, \beta < \kappa^+ \) such that \( t^\mathcal{A}_0(\bar{a}_{i_0}/\mathcal{A}_0) = t^\mathcal{A}_0(\bar{a}_{i_3}/\mathcal{A}_0) \). Then by stationarity, also \( t^\mathcal{A}(\bar{a}_{i_0}/\mathcal{A}) = t^\mathcal{A}(\bar{a}_{i_3}/\mathcal{A}) \).

We note that under tameness this result implies also stability respect to Galois types, see Theorem 4.11.
Our first result considering saturated models says that there always are weakly saturated models of size greater or equal to the Hanf number. The extension property is not needed to show this.

**Theorem 4.7.** Let \( (\mathbb{K}, \preceq_{\mathbb{K}}) \) be an \( \aleph_0 \)-stable finitary abstract elementary class and suppose that \( \lambda \geq \aleph_0 \). Then there is a weakly saturated model of size \( \lambda \).

**Proof.** Since for each infinite \( A \), the number of weak types over \( A \) is \( |A| \), there is an increasing sequence \( (\mathcal{A}_i)_{i<\lambda} \) of \( \aleph_0 \)-saturated models such that \( |\mathcal{A}_i| < \lambda \) for all \( i < \lambda \cdot \lambda \) and every weak type over \( \mathcal{A}_i \) is realized in \( \mathcal{A}_{i+1} \). Let \( \mathcal{A} = \bigcup_{i<\lambda} \mathcal{A}_i \). We claim that \( \mathcal{A} \) is as wanted. For this let \( A \subseteq \mathcal{A} \) be of power \( < \lambda \) and \( a \) be any finite sequence of elements of the monster model. We need to find \( b \in \mathcal{A} \) such that \( \text{tp}(b/A) = \text{tp}(a/A) \). If \( \lambda \) is regular then the existence of \( b \) follows immediately from the construction. So we assume that \( \lambda \) is a limit cardinal and \( \lambda > \aleph_0 \).

Let \( \gamma < \lambda \cdot \lambda \) be such that \( a \not\equiv_{\mathcal{A}_\gamma} \mathcal{A} \). Let \( \alpha = \gamma + \aleph_0 \cdot |A| \cdot \lambda < \lambda \cdot \lambda \). Since for each \( \bar{c} \in A \), there is \( i < \alpha \) such that \( \bar{c} \not\equiv_{\mathcal{A}_i} \mathcal{A} \) and \( cf(\alpha) > |A| \), there is \( \gamma \leq \beta < \alpha \) such that for each \( c \in A \), \( \bar{c} \not\equiv_{\mathcal{A}_\beta} \mathcal{A}\).

For all \( \beta \leq i < \alpha \), choose \( \bar{b}_i \in \mathcal{A}_{i+1} \) such that \( \text{tp}(\bar{b}_i/A_i) = \text{tp}(a/A_i) \). If for some \( \bar{b}_i \), \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta} \mathcal{A} \), as above, we have found \( b \). So we may assume that for all \( \bar{b}_i \) there is \( \bar{c}_i \in A \) such that \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta, \bar{c}_i} \mathcal{A} \). By the pigeon hole principle we may assume that there is \( \bar{c}_i \in A \) such that for all \( \bar{b}_i \), \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta} \mathcal{A} \). Choose a countable \( \aleph_0 \)-saturated \( \mathcal{B}_\alpha \subseteq \mathcal{A}_\beta \) such that \( \bar{a} \not\equiv_{\mathcal{B}_\alpha} \mathcal{A} \) and \( \bar{c} \not\equiv_{\mathcal{B}_\alpha} \mathcal{A} \). Since \( \text{tp}(\bar{b}_i/A\bar{b}_i) = \text{tp}(\bar{b}_j/A\bar{b}_j) \), \( \bar{b}_i, \bar{b}_j \in \mathcal{A}_\alpha \) and \( \text{tp}(\bar{c}/\mathcal{A}_\alpha) \) does not split over any finite \( E \subseteq \mathcal{A}_\alpha \), we get that \( \text{tp}(\bar{b}_i/E \cup \bar{c}) = \text{tp}(\bar{b}_j/E \cup \bar{c}) \) for any finite \( E \subseteq \mathcal{A}_\beta \). Thus for each \( \beta < i < j < \alpha \), \( \text{tp}(\bar{b}_i/A\bar{b}_i \cup \bar{c}) = \text{tp}(\bar{b}_j/A\bar{b}_j \cup \bar{c}) \).

For a fixed \( i \), since \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta} \mathcal{A} \), by monotonicity \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta} \mathcal{A} \cup \bar{c} \) and from finite character of splitting we get some countable \( \aleph_0 \)-saturated \( \mathcal{B} \subseteq \mathcal{A}_\beta \) such that \( \bar{b}_i \not\equiv_{\mathcal{A}_\beta} \mathcal{B} \cup \bar{c} \). Since \( \bar{b}_i \not\equiv_{\mathcal{B}} \mathcal{B} \), we get from transitivity that \( \bar{b}_i \not\equiv_{\mathcal{B}} \bar{c} \). Since all \( \bar{b}_i \)'s have the same weak type over \( \bar{c} \cup \mathcal{B} \), this holds for all \( \bar{b}_i \), \( \beta < i < \alpha \).

Now for all \( \beta < i < \alpha \), choose \( \bar{c}_i \in \mathcal{A}_{i+1} \) such that \( \text{tp}(\bar{c}_i/A_i) = \text{tp}(\bar{c}/A_i) \). Then

(a) if \( i > j \), \( \bar{b}_i \not\equiv_{\mathcal{A}_{i+1}} \mathcal{B} \) by the choice of \( \bar{b}_i \) and \( \mathcal{B} \)

\((\text{Since } \bar{a} \not\equiv_{\mathcal{A}_{i+1}} \mathcal{A}_{i+1} \text{ and } \text{tp}(\bar{b}_i/A_{i+1}) = \text{tp}(\bar{a}/A_{i+1}))\).

(b) if \( i < j \), \( \bar{b}_i \not\equiv_{\mathcal{A}_{i+1}} \mathcal{B} \) by the choice of \( \bar{c}_j \)

\((\text{Since } \bar{b}_i \not\equiv_{\mathcal{A}_{i+1}} \mathcal{B} \text{ and } \text{tp}(\bar{c}_j/\mathcal{B} \cup \bar{b}_i) = \text{tp}(\bar{c}_j/\mathcal{B} \cup \bar{b}_i))\).
Since $|\alpha \setminus \beta| \geq \aleph_0$, as in the proof of Theorem 4.3, we get a contradiction with $\aleph_0$-stability.

As a corollary we get the following.

**Corollary 4.8.** Let $(\mathcal{K}, \leq_{\mathcal{K}})$ be a finitary AEC, stable in $\aleph_0$, and let $\lambda \geq \aleph_0$. Then there is an $\aleph_1$-saturated model of size $\lambda$.

Our second result shows that if $(\mathcal{K}, \leq_{\mathcal{K}})$ has in addition the existence property, we have a weakly saturated model in every infinite cardinal. The proof uses symmetry.

**Theorem 4.9.** Assume that $(\mathcal{K}, \leq_{\mathcal{K}})$ is a finitary AEC, stable in $\aleph_0$ and has the extension property. Then there is a weakly saturated model in every infinite cardinal $\lambda$.

**Proof.** Clearly we may assume that $\lambda > \aleph_0$. As in the previous proof, we have an increasing sequence $(\mathcal{A}_i)_{i<\lambda}$ of $\aleph_0$-saturated models such that $|\mathcal{A}_i| < \lambda$ for all $i < \lambda$ and every weak type over $\mathcal{A}_i$ is realized in $\mathcal{A}_{i+1}$. Again we claim that $\mathcal{A} = \bigcup_{i<\lambda} \mathcal{A}_i$ is as wanted.

For this, we take $A \subset \mathcal{A}$ of size strictly less than $\lambda$ and some $\bar{a} \in \mathcal{M}$. We may assume that $\lambda$ is a limit cardinal. Let $\gamma < \lambda$ be such that $\bar{a} \downarrow^{s}_{\mathcal{A}_\gamma} \mathcal{A}$, and let $\alpha = \gamma + |A|^+ < \lambda$. As in the previous proof, since $cf(\alpha) > |A|$, there is $\gamma < \beta < \alpha$ such that $\bar{c} \downarrow^{s}_{\mathcal{A}_\beta} \mathcal{A}_\alpha$ for each $\bar{c} \in A$.

Choose $\bar{b} \in \mathcal{A}_{\beta+1}$ such that $tp^w(\bar{b}/\mathcal{A}_\beta) = tp^w(\bar{a}/\mathcal{A}_\beta)$. Since $\bar{b} \in \mathcal{A}_\alpha$ and $\bar{c} \downarrow^{s}_{\mathcal{A}_\beta} \mathcal{A}_\alpha$ for all $\bar{c} \in A$, by symmetry and finite character of splitting, $\bar{b} \downarrow^{s}_{\mathcal{A}_\beta} A$. By stationarity, $tp^w(\bar{b}/A) = tp^w(\bar{a}/A)$.

**4.3. Tameness.** Tameness is one property that implies the extension property. Both of our main examples, excellent classes and $\aleph_0$-stable homogeneous classes, have it. We could also take tameness as one of our assumptions for a finitary abstract elementary class, but we choose not to do so, since the assumption would be really needed only to prove that the class has the extension property. We also think that tameness as an assumption seems quite strong, since it considers such a complicated concept as the automorphism group of the monster model. Extension property for weak types has a more local nature, and also $\aleph_0$-extension is usually relatively easy to check in particular examples. The study of tame $\aleph_0$-stable finitary AEC’s will also give a nice theory, due to the fact that Galois types and weak types will then agree over arbitrary models, see Theorem 4.11.

**Definition 4.10 (Tameness).** Let $LS(\mathcal{K}) \leq \kappa \leq \lambda$. We say that $(\mathcal{K}, \leq_{\mathcal{K}})$ is $(\kappa, \lambda)$-tame, if for each model $\mathcal{A}$ of size $\lambda$ such that

$$tp^g(\bar{a}/\mathcal{A}) \neq tp^g(\bar{b}/\mathcal{A}),$$


there is $\mathcal{B} \leq_{K} \mathcal{A}$ of size $\kappa$ such that
\[ \text{tp}^g(\bar{a}/\mathcal{B}) \neq \text{tp}^g(\bar{b}/\mathcal{B}). \]
We say that $(K, \leq_{K})$ is tame if it is $(LS(K), \lambda)$-tame for each cardinal $\lambda \geq LS(K)$.

Simply, if $(K, \leq_{K})$ is tame and $\mathcal{A}$ is a model, $\text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A})$ if and only if $\text{tp}^g(\bar{a}/\mathcal{B}) = \text{tp}^g(\bar{b}/\mathcal{B})$ for every countable $\mathcal{B} \leq_{K} \mathcal{A}$. The next result follows from Theorem 3.12.

**Theorem 4.11.** Assume that $(K, \leq_{K})$ is $\aleph_0$-stable and tame finitary AEC. If $\mathcal{A}$ is a model, we have that $\text{tp}^u(\bar{a}/\mathcal{A}) = \text{tp}^u(\bar{b}/\mathcal{A})$ if and only if $\text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A})$.

This gives an improvement of tameness: If $\mathcal{A}$ is a model and $\text{tp}^g(\bar{a}/\mathcal{A}) \neq \text{tp}^g(\bar{b}/\mathcal{A})$, then there is finite $A_0 \subset \mathcal{A}$ such that $\text{tp}^g(\bar{a}/A_0) \neq \text{tp}^g(\bar{b}/A_0)$.

Now similarly as in 3.13, only by induction on $|\mathcal{A}|$, using Theorem 4.11 instead of Theorem 3.12, we can prove the following lemma. In homogeneous model theory, this property is sometimes called weak compactness, only there we need not assume that $\mathcal{A}$ is a model.

**Lemma 4.12.** Assume $(K, \leq_{K})$ is tame, finitary and stable in $\aleph_0$. Let $\mathcal{A}$ be a model, and suppose that for each finite $A \subset \mathcal{A}$ there is $\bar{a}_A$ such that if $B \subset A$, then $\text{tp}^g(\bar{a}_B/B) = \text{tp}^g(\bar{a}_A/B)$. Then there is $\bar{a}$ such that for each finite $A \subset \mathcal{A}$,
\[ \text{tp}^g(\bar{a}/A) = \text{tp}^g(\bar{a}_A/A). \]

As a corollary we get that a tame, finitary and $\aleph_0$-stable $(K, \leq_{K})$ has $(\lambda, \kappa)$-local and $(\lambda, \kappa)$-compact Galois types in the sense of [1] for every $\lambda$ and $\kappa$. Even better, any increasing chain of types over sets (not necessarily models) $A_i$, $i < \kappa$, such that $\bigcup_{i<\kappa} A_i$ is a model, has a unique realization. Now we get the full extension property as in Remark 3.19.

**Theorem 4.13** (Extension property). Assume that $(K, \leq_{K})$ is $\aleph_0$-stable and tame finitary AEC. Let $\mathcal{A}$ be $\aleph_0$-saturated model and $E \subset \mathcal{A}$ finite such that $\text{tp}^u(\bar{a}/\mathcal{A})$ does not split over $E$. Then if $B \supset \mathcal{A}$, there is $\bar{b}$ such that $\text{tp}^u(\bar{a}/\mathcal{A}) = \text{tp}^u(\bar{b}/\mathcal{A})$ and $\text{tp}^u(\bar{b}/B)$ does not split over $E$.

As a corollary we get the following.

**Corollary 4.14.** Assume that $(K, \leq_{K})$ is a tame $\aleph_0$-stable finitary AEC. Then $(K, \leq_{K})$ has a notion of splitting with the following properties:

1. **Invariance** If $f$ is an automorphism of $\mathfrak{M}$, $\bar{a} \forces_{A} B$ if and only if $f(\bar{a}) \forces_{f(A)} f(B)$.
2. **Monotonicity** If $A \subset B \subset C \subset D$ and $\bar{a} \forces_{A} D$, then $\bar{a} \forces_{B} C$. 
(3) **Finite character** Let $\mathcal{A}$ be an $\aleph_0$-saturated model and $\mathcal{A} \subset B$. Then $\bar{a} \downarrow^*_\mathcal{A} B$ if and only if $\bar{a} \downarrow^*_\mathcal{A} B_0$ for every finite $B_0 \subset B$.

(4) **Local character** For each model $\mathcal{A}$ and finite sequence $\bar{a}$ there is finite $E \subset \mathcal{A}$ such that $\bar{a} \downarrow^*_E \mathcal{A}$.

(5) **Extension** Let $\mathcal{A}$ be an $\aleph_0$-saturated model and $\mathcal{A} \subset B$. For each $\bar{a}$ there is $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\bar{b} \downarrow^*_\mathcal{A} B$. Moreover, if $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over the finite subset $E$, then $\text{tp}^w(\bar{b}/B)$ does not split over the finite set $E$.

(6) **Transitivity** If $A \subset B \subset C$ and $\mathcal{B}$ is an $\aleph_0$-saturated model, then $\bar{a} \downarrow^*_A C$ if and only if $\bar{a} \downarrow^*_B \mathcal{B}$ and $\bar{a} \downarrow^*_B C$.

(7) **Symmetry** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. Then $\bar{a} \downarrow^*_\mathcal{A} \bar{b}$ if and only if $\bar{b} \downarrow^*_\mathcal{A} \bar{a}$.

(8) **Stationarity** Assume $\mathcal{A}$ is an $\aleph_0$-saturated model and $A \subset B$. If $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow^*_A B$ and $\bar{b} \downarrow^*_B B$, then $\text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B)$.

We remark that if $(\mathbb{K}, \preceq)$ is a tame $\aleph_0$-stable finitary AEC, Theorems 4.9, 4.11 and 4.13 imply that there is also a saturated model in each infinite cardinal respect to Galois types, not only weak types. To emphasize this, we state it as a theorem.

**Theorem 4.15.** Assume that $(\mathbb{K}, \preceq)$ is a tame $\aleph_0$-stable finitary AEC and $\lambda$ is infinite. Then there is a Galois saturated model of size $\lambda$.

### 4.4. Categoricity.

Another theorem tells us that extension property follows also from $\kappa$-categoricity for suitable $\kappa$.

**Definition 4.16.** We say that $(\mathbb{K}, \preceq)$ is $\kappa$-categorical, if whenever $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ and $|\mathcal{A}| = |\mathcal{B}| = \kappa$, then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Another theorem of Shelah’s tells us that $\aleph_0$-stability is implied by $\kappa$-categoricity for any uncountable $\kappa$, see for example [1] for the proof.

**Theorem 4.17.** (Shelah) Let $(\mathbb{K}, \preceq)$ be a finitary abstract elementary class, which is $\kappa$-categorical for some uncountable $\kappa$. Then it is $\aleph_0$-stable.

For convenience we define $\lambda$-dense to be the concept that is usually called $\lambda$-dense without endpoints.

**Definition 4.18.** Let $(I, \prec)$ be a linear ordering and $C, D \subset I$. When $c < d$ for each $c \in C, d \in D$, we denote $C < D$. We say that $(I, \prec)$ is $\lambda$-dense, if for each $C, D \subset I$, $|C|, |D| < \lambda$ and $C < D$, there is $i \in I$ such that $C < \{i\} < D$, and for each $C \subset I$, $|C| < \lambda$, there are $i, j \in I$ such that $\{i\} < C < \{j\}$.

We say that $(I, \prec)$ is dense, if it is $\aleph_0$-dense.
The idea of the argument in 4.19 is also due to Shelah, see Lemma 6.3 of [17] or [1] for a simpler proof. For completeness, we sketch the proof here.

**Proposition 4.19.** Let $(K, \leq_K)$ be categorical in $\kappa$ and assume that there is an $\aleph_1$-saturated model of size $\kappa$. Then if $\mathcal{A}$ is an $\aleph_0$-saturated model, $\mathcal{A} \subset B$, $|B| \leq \kappa$ and $\text{tp}^w(\bar{a}/\mathcal{A})$ does not split over finite $E \subset \mathcal{A}$, there is $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A})$ such that $\text{tp}^w(\bar{b}/B)$ does not split over $E$.

**Proof.** By Theorem 4.17, $(K, \leq_K)$ is also $\aleph_0$-stable. By stationarity, we may assume that $\mathcal{A}$ is countable. From 2.22 we get that there is a countable $\tau^*$-order-indiscernible sequence in $\mathcal{M}$, and from 2.23 also a $\tau^*$-order-indiscernible $(I, <)$, where $I \subset \mathcal{M}$, $|I| = \kappa$ and $(I, <)$ is a dense linear order. Let $SH(J)$ denote the closure of $J \subset I$ with $\tau^*$. The set $B \cup \{\bar{a}\}$ is included in $\mathcal{B}$ for some model $\mathcal{B}$ of size $\kappa$. From $\kappa$-categoricity we get that $\mathcal{B}$ and $SH(I)$ are isomorphic. Thus we may assume that $\mathcal{B} \cup \{\bar{a}\} \subset SH(I)$. We have that $B \subset SH(K)$ for some $K \subset I$ such that $|K| = \lambda$. We assumed that $\mathcal{A}$ is countable, and thus $\mathcal{A} \leq_k SH(J)$ for some countable $J \subset I$. Since $I$ is dense, we may assume that $(J, <) \cong (\mathbb{Q}, <)$ and we may also assume that $SH(J, <)$ is $\aleph_0$-saturated. By countable extension there is $\bar{a}' \in \mathcal{M}$ such that $\text{tp}^w(\bar{a}'/\mathcal{A}) = \text{tp}^w(\bar{a}/\mathcal{A})$ and $\text{tp}^w(\bar{a}'/SH(J))$ does not split over $E$. Since $SH(I)$ is $\aleph_1$-saturated by categoricity, we may assume that $\bar{a}'$ is in $SH(I)$.

By Lemma 2.23 we have that $I$ is a suborder of a $\tau^*$-order-indiscernible $\lambda^+$-dense linear order in $\mathcal{M}^*$. We call this order $I^+$. Let $i_0 < ... < i_{n-1} \in I$ and functions $F_{k_0}^{n_0}, ..., F_{k_p}^{n_p} \in \tau^*$ be such that

\begin{equation}
\bar{a}' = ((F_{k_0}^{n_0}(i_0^0, ..., i_{n_0-1}^0), ..., (F_{k_p}^{n_p}(i_0^p, ..., i_{n_p-1}^p)) \end{equation}

and $\{i_0, ..., i_{n-1}\} = \{i_0^0, ..., i_{n_0-1}^0, ..., i_0^p, ..., i_{n_p-1}^p\}$. By $\lambda^+$-density of $I^+$ and density of $J$ we find $j_0 < ... < j_{n-1} \in I^+$ such that

1. if $i_k \in J$, then $j_k = i_k$,
2. $i_k < j$ if and only if $j_k < j$ for each $j \in J$,
3. if $i_k \notin J$, then $j_k \notin J \cup K$,
4. if there is $k \in K \setminus J$ such that $j_k < k < j_{k+1}$, then there are infinitely many $j \in J$ such that $j_k < j < j_{k+1}$,
5. if there is $k \in K$ between some $j_k$ and $j \in J$, then there are infinitely many $j \in J$ in that same interval,
6. if there are $k \in K$ such that $k < j_0$, then there is infinitely many such $j \in J$ and similarly for $k > j_{n-1}$.

Finally let $\bar{b}$ be generated from $j_0, ..., j_{n-1}$ as $\bar{a}'$ was from $i_0, ..., i_{n-1}$, that is

$$\bar{b} = ((F_{k_0}^{n_0}(j_0^0, ..., j_{n_0-1}^0), ..., (F_{k_p}^{n_p}(j_0^p, ..., j_{n_p-1}^p)),$$
where \(j^r_{qs} = j_k\) if and only if \(j^r_{qs} = i_k\) in 4. Now for every finite \(J_0 \subset J\) there is order-preserving \(f\) mapping \(j_k\) to \(i_k\) for every \(0 \leq k < n\) such that \(f \upharpoonright J_0 = \text{id}_{J_0}\). Also for every finite \(K_0 \subset K \setminus J_0\) we can extend this mapping such that it maps \(K_0\) to \(J\). Since every order-preserving partial map \(f : I^+ \rightarrow I^+\) extends to an automorphism \(F \in \text{Aut}({\mathfrak M}^+),\) we get that \(\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})\) and \(\text{tp}^w(\bar{b}/B)\) does not split over \(E\). □

If \((\mathbb{K}, \leq_{\mathbb{K}})\) is categorical in some \(\lambda \geq \mathbb{H}\), we get by Corollary 4.8 that there exists an \(\aleph_1\)-saturated model in \(\lambda\), and then by Proposition 4.19 that \((\mathbb{K}, \leq_{\mathbb{K}})\) has the \(\mathbb{H}\)-extension property. Hence by Proposition 4.4 \((\mathbb{K}, \leq_{\mathbb{K}})\) has the \(\lambda\)-extension property for all \(\lambda\). We state this as a theorem.

**Theorem 4.20 (Extension property).** Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is finitary and categorical in some \(\kappa \geq \mathbb{H}\). Then \((\mathbb{K}, \leq_{\mathbb{K}})\) has the \(\lambda\)-extension property for all \(\lambda\).

As a corollary we get the following.

**Corollary 4.21.** Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is a finitary AEC, categorical in some \(\kappa \geq \mathbb{H}\). Then \((\mathbb{K}, \leq_{\mathbb{K}})\) has a notion of splitting with the properties listed in Corollary 4.14.

We remark that by Theorems 4.9 and 4.20, categoricity above the Hanf number implies that there are weakly saturated models in each infinite cardinal.

## 5. Strong splitting

In this section we assume \((\mathbb{K}, \leq_{\mathbb{K}})\) to be a finitary abstract elementary class, stable in \(\aleph_0\) and has the extension property. Chapters 5 and 6 follow closely the paper [7]. We will replace the notion of strong indiscernibility defined in [7] with a slightly weaker notion (a priori), but this will not affect most of the proofs.

We define a Morley sequence using weak types instead of Galois types.

**Definition 5.1 (Morley sequence).** Suppose \(\mathcal{A}\) is an \(\aleph_0\)-saturated model. We say that \((\bar{a}_i)_{i<\alpha}\) is a Morley sequence over \(\mathcal{A}\) if for each \(i < j < \alpha\), \(\text{tp}^w(\bar{a}_i/\mathcal{A}) = \text{tp}^w(\bar{a}_j/\mathcal{A})\) and for each \(i < \alpha\), \(\bar{a}_i \downarrow_{\mathcal{A}} \bigcup_{j<i} \bar{a}_j\).

**Lemma 5.2.** Let \((\bar{a}_i)_{i<\alpha}\) be a Morley sequence over a countable \(\aleph_0\)-saturated model \(\mathcal{A}\). Then for every \(n\) and \(i_0 < ... < i_n < \alpha\) we have that \(\text{tp}^g(\bar{a}_{i_0},...,a_n/\mathcal{A}) = \text{tp}^g(\bar{a}_{i_0},...,\bar{a}_{i_n}/\mathcal{A})\).

**Proof.** Since \(\mathcal{A}\) is a countable model, by Theorem 3.12 it is enough to show that \(\text{tp}^w(\bar{a}_{i_0},...,a_n/\mathcal{A}) = \text{tp}^w(\bar{a}_{i_0},...,\bar{a}_{i_n}/\mathcal{A})\). We do the proof by induction on \(n\). When \(n = 0\), the claim follows from the definition. Assume that the claim holds for all
$i_0, \ldots, i_{m-1} < \alpha$. If $i_m = m$, the claim is trivial. Assume that $i_m > m$ and let $E \subset \mathcal{A}$ be finite such that $\text{tp}^w(\bar{a}_{i_m}/\mathcal{A} \cup \{a_i : i < i_m\})$ does not split over $E$ and $\text{tp}^w(\bar{a}_m/\mathcal{A} \cup \{a_i : i < m\})$ does not split over $E$. Then let $C \subset \mathcal{A}$ be finite.

Since $\mathcal{A}$ is $\aleph_0$-saturated, we have an automorphism $f \in \text{Aut}(\mathcal{M}/E \cup C)$ such that $f(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{m-1}}) \in \mathcal{A}$. Furthermore from induction we get $f' \in \text{Aut}(\mathcal{M}/E \cup C)$ such that $f'(\bar{a}_0, \ldots, \bar{a}_{m-1}) = (\bar{a}_{i_0}, \ldots, \bar{a}_{i_{m-1}})$. Then $(f \circ f')(\bar{a}_0, \ldots, \bar{a}_{m-1}) = f(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{m-1}})$ and $f \circ f'$ is in $\text{Aut}(\mathcal{M}/(E \cup C))$, and from the choice of $E$ we get $f_1 \in \text{Aut}(\mathcal{M}/C \cup \{\bar{a}_m\})$

such that $f_1(\bar{a}_0, \ldots, \bar{a}_{m-1}) = f(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{m-1}})$. Similarly we get $f_2 \in \text{Aut}(\mathcal{M}/C \cup \{\bar{a}_m\})$ such that $f_2(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{m-1}}) = f(\bar{a}_0, \ldots, \bar{a}_{i_{m-1}})$. Then since $\text{tp}^w(\bar{a}_m/\mathcal{A}) = \text{tp}^w(\bar{a}_{i_m}/\mathcal{A})$, we also have

$$h \in \text{Aut}(\mathcal{M}/C \cup \{f(\bar{a}_{i_0}), \ldots, f(\bar{a}_{i_{m-1}})\})$$

such that $h(\bar{a}_m) = \bar{a}_{i_m}$. Finally $(f_1^{-1} \circ h \circ f_2)(\bar{a}_0, \ldots, \bar{a}_m) = (\bar{a}_0, \ldots, \bar{a}_m)$ and $(f_1^{-1} \circ h \circ f_2)$ is in $\text{Aut}(\mathcal{M}/C)$.\]

We note that under tameness the above Lemma holds not only for a Morley sequence over a countable $\aleph_0$-saturated model $\mathcal{A}$, but for an $\aleph_0$-saturated model $\mathcal{A}$ of arbitrary size.

**Lemma 5.3.** Let $E$ be a set, $I$ a sequence of tuples such that $i \neq j$ implies $\bar{a}_i \neq \bar{a}_j$, and $\aleph_0 + |E| \leq \lambda < |I|$. Then there is an $\aleph_0$-saturated model $\mathcal{A}$ of size $\lambda$ containing $E$ and subsequence $(\bar{a}_i)_{i < \lambda^+} \subset I$ that is a Morley sequence over $\mathcal{A}$.

**Proof.** Construct an $<_\aleph$-increasing chain of $\aleph_0$-saturated models $E \subset \mathcal{A}_i$, $i < \lambda^+$, of size $\lambda$ such that for limit $i$, $\mathcal{A}_i = \bigcup_{j < i} \mathcal{A}_j$ and a subsequence $(\bar{a}_i)_{i < \lambda^+} \subset I$ such that $\bar{a}_i \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$. This is possible since $|I| \geq \lambda^+$. For each $i$ there is some finite $E_i \subset \mathcal{A}_i$ such that $\text{tp}^w(\bar{a}_i/\mathcal{A}_i)$ does not split over $E_i$. We may define $f : \lambda^+ \rightarrow \lambda^+$ such that

$$f(i) = \min\{j \leq i : \exists \text{ finite } E \subset \mathcal{A}_j \text{ s.t. } \text{tp}^w(\bar{a}_i/\mathcal{A}_i) \text{ does not split over } E)\}.$$

We note that $f$ is a decreasing function and that $f$ is strictly decreasing on a stationary set $\{\alpha < \lambda^+ : \alpha \text{ is a limit ordinal}\}$. Then by Fodor’s Lemma we may find a stationary $S \subset \lambda^+$ such that $f$ is constant on $S$, say $f(i) = i_0$ for all $i \in S$. When $i \in S$, there is a finite $E_i \subset \mathcal{A}_{i_0}$ such that $\text{tp}^w(\bar{a}_i/\mathcal{A}_i)$ does not split over $E_i$. Since there are only $\lambda$-many finite subsets in $\mathcal{A}_{i_0}$, by the pigeonhole principle there is a subsequence $(\mathcal{A}_{j_i})_{i < \lambda^+}$ such that $\text{tp}^w(\bar{a}_{j_i}/\mathcal{A}_{j_i})$ does not split over the same finite
$E' \subseteq A$. Then let $\mathcal{A}' \subseteq \mathcal{A}_0$ be countable such that $E' \subseteq \mathcal{A}'$. By $\aleph_0$-stability we may choose again a subsequence $(\mathcal{A}_{k_i})_{i<\lambda^+}$ such that $\text{tp}^w(\bar{a}_{k_i}/\mathcal{A}') = \text{tp}^w(\bar{a}_{k_j}/\mathcal{A}')$ for each $k_i, k_j < \lambda^+$. Since $E' \subseteq \mathcal{A}'$, we have that $\bar{a}_{k_i} \downarrow_{\mathcal{A}'} \bar{a}_{k_j}$, and thus also get from stationarity that actually $\text{tp}^w(\bar{a}_{k_i}/\mathcal{A}_0) = \text{tp}^w(\bar{a}_{k_j}/\mathcal{A}_0)$ for each $k_i, k_j < \lambda^+$. Denote $\mathcal{A} = \mathcal{A}_0$. From monotonicity we get that $\bar{a}_{k_i} \downarrow_{\mathcal{A}} \bigcup_{j<i} \{\bar{a}_{k_j}\}$ for each $k_i < \lambda^+$. Now $(\bar{a}_{k_i})_{i<\lambda^+}$ is a Morley sequence over $\mathcal{A}$, $E \subseteq \mathcal{A}$ and $\{\bar{a}_{k_i} : i < \lambda\} \subseteq I$. \qed

Now we will introduce our notion of strong indiscernibility. In [7] the notion is similar except that we are able to extend an arbitrary partial function $f : \lambda \rightarrow \lambda$ to automorphism. Here we are only able to extend an order-preserving one.

**Definition 5.4 (Strong indiscernibility).** We say that a sequence $(\bar{a}_i)_{i<\alpha}$ of distinct tuples is strongly indiscernible over $E$, or strongly $E$-indiscernible, if for every ordinal $\lambda \geq \alpha$ there is a sequence $(\bar{a}_i)_{i<\lambda}$ extending $(\bar{a}_i)_{i<\alpha}$ such that for any order-preserving partial $f : \lambda \rightarrow \lambda$, there is $F \in \text{Aut}(\mathcal{M}/E)$ such that $F(\bar{a}_i) = \bar{a}_{f(i)}$ for all $i \in \text{dom}(f)$.

**Remark 5.5.** If $(\bar{a}_i)_{i<\alpha}$ is a strongly indiscernible sequence over $E$ and $f \in \text{Aut}(\mathcal{M}/E)$, then also $(f(\bar{a}_i))_{i<\alpha}$ is strongly indiscernible over $E$.

The next remark follows from the the homogeneity of $\mathcal{M}^*$.

**Remark 5.6.** If a sequence $(\bar{a}_i)_{i<\alpha}$, $\alpha$ infinite, is $\tau^*$-order-indiscernible over $E$, then it is strongly $E$-indiscernible.

In the context of [7] it holds that a sequence is strongly indiscernible over $E$ if and only if it is a Morley sequence over some model containing $E$. Here we state an analogous result for countable $E$.

**Definition 5.7.** We say that two sequences $(\bar{a}_i)_{i<\alpha}$ and $(\bar{b}_j)_{j<\beta}$ are equivalent over $E$, if for every finite $n$ we have that $\text{tp}^g(\bar{a}_0, \ldots, \bar{a}_n/E) = \text{tp}^g(\bar{b}_0, \ldots, \bar{b}_n/E)$.

**Lemma 5.8.** Let $E$ be countable and $(\bar{a}_i)_{i<\omega}$ a sequence of finite tuples. The following are equivalent:

1. The sequence $(\bar{a}_i)_{i<\omega}$ is $E$-equivalent to a sequence $(\bar{b}_i)_{i<\omega}$, which is a Morley sequence over some countable $\aleph_0$-saturated model $\mathcal{A}$ containing $E$.
2. The sequence $(\bar{a}_i)_{i<\omega}$ is $E$-equivalent to a sequence $(\bar{b}_i)_{i<\omega}$, which is $\tau^*$-order-indiscernible over $E$.
3. The sequence $(\bar{a}_i)_{i<\omega}$ is $E$-equivalent to a strongly $E$-indiscernible sequence $(\bar{b}_i)_{i<\omega}$.

**Proof.** First we prove that (2) follows from (1). Let $(\bar{a}_i)$ be $E$-equivalent to a sequence $(\bar{b}_i)_{i<\omega}$, which is a Morley sequence over a countable $\aleph_0$-saturated model
sequence over the model $M$. Then for any saturated $(\bar{\alpha}) \subset \mathfrak{A}$.

**Corollary 5.9.** Let $E$ be countable and $I$ an uncountable sequence of distinct tuples. Then for any $n < \omega$ there is a subsequence $(\bar{a}_0, \ldots, \bar{a}_{n-1}) \subset I$, which is the beginning of a strongly $E$-indiscernible sequence.

**Proof.** Let $n < \omega$ be given. By Lemma 5.3, there is a countable $\aleph_0$-saturated model $\mathfrak{A}$ containing $E$ and a subsequence $(\bar{a}_i)_{i<\omega} \subset I$ such that it is a Morley sequence over $\mathfrak{A}$. Furthermore, by Lemma 5.8, there is a strongly $E$-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ such that $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_i)_{i<\omega}$ are $E$-equivalent. Thus we have $f \in \text{Aut}(\mathfrak{M}/E)$ such that $f(\bar{b}_i) = \bar{a}_i$ for $i < n$. Now we have that $(f(\bar{b}_i))_{i<\omega}$ is also strongly $E$-indiscernible, and it extends $(\bar{a}_0, \ldots, \bar{a}_{n-1})$.

The following technical lemmas are needed in the next sections.

**Lemma 5.10.** Let $E \subset B$ with $B$ finite. Let $(\bar{a}_i)_{i<\omega}$ be strongly $E$-indiscernible such that for any $i_0 < \ldots < i_n < \omega_1$, $\text{tp}^g(\bar{a}_0, \ldots, \bar{a}_n/B) = \text{tp}^g(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/B)$. Then, for each $n < \omega$, there exists a strongly $B$-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_i' = \bar{a}_i$ for each $i < n$.

**Proof.** Let $n < \omega$ be given. By Corollary 5.9 there is a subsequence $(\bar{a}_{k_0}, \ldots, \bar{a}_{k_{n-1}}) \subset (\bar{a}_i)_{i<\omega}$ such that it is the beginning of a strongly $B$-indiscernible sequence $(\bar{b}_i)_{i<\omega}$. Thus we have that

$$\text{tp}^g(\bar{a}_0, \ldots, \bar{a}_{n-1}/B) = \text{tp}^g(\bar{a}_{k_0}, \ldots, \bar{a}_{k_{n-1}}/B) = \text{tp}^g(\bar{b}_0, \ldots, \bar{b}_{n-1}/B).$$

When $f \in \text{Aut}(\mathfrak{M})$ is such that $f \upharpoonright B = \text{id}_B$ and $f(\bar{b}_i) = \bar{a}_i$ for $i < n$, the sequence $(f(\bar{b}_i))_{i<\omega}$ is also strongly $B$-indiscernible. This sequence is as we wanted.
Lemma 5.11. Let $E$ be finite. There exists an $\aleph_0$-saturated model $\mathcal{A}$ containing $E$ and a set $\mathcal{I}$ of strongly $E$-indiscernible sequences $(\bar{a}_i)_{i<\omega_1} \subset \mathcal{A}$ with the following properties:

1. If $(\bar{a}_i)_{i<\omega_1}$ is included in some sequence in $\mathcal{I}$, then $(\bar{a}_i)_{i<\omega_1} \in \mathcal{I}$.
2. Whenever $(\bar{a}_i)_{i<\omega}$ is strongly $E$-indiscernible with $\bar{a}_0, \bar{a}_1 \in \mathcal{A}$, there is a strongly $E$-indiscernible sequence $(\bar{a}'_i)_{i<\omega_1} \in \mathcal{I}$ such that $\bar{a}'_0 = \bar{a}_0$ and $\bar{a}'_1 = \bar{a}_1$.
3. For every $(\bar{a}_i)_{i<\omega_1} \in \mathcal{I}$ and $k < \omega_1$ there is $f \in \text{Aut}(\mathcal{A})$ such that $f \upharpoonright E = \text{id}_E$ extending the mapping $\bar{a}_i \mapsto \bar{a}_{i+k}$ for each $i < \omega_1$.

Proof. Let $\lambda = 2^{\aleph_1}$. We construct an increasing sequence of $\aleph_0$-saturated models $\mathcal{A}_n$ for $n < \omega$, an increasing sequence of sets $\mathcal{I}_n$ of strongly $E$-indiscernible sequences and choose for each $I \in \mathcal{I}_n$ and $k < \omega_1$ an automorphism $f_k \in \text{Aut}(\mathcal{M}/E)$ such that:

1. For $n < \omega$, $\bigcup \mathcal{I}_n \subset \mathcal{A}_n$, $|\mathcal{A}_n| \leq \lambda$ and $|\mathcal{I}_n| \leq \lambda$.
2. Whenever $(\bar{a}_i)_{i<\omega}$ is strongly $E$-indiscernible with $\bar{a}_0, \bar{a}_1 \in \mathcal{A}_n$, there is a strongly $E$-indiscernible sequence $(\bar{a}'_i)_{i<\omega_1} \in \mathcal{I}_{n+1}$ such that $\bar{a}'_0 = \bar{a}_0$ and $\bar{a}'_1 = \bar{a}_1$.
3. If $(\bar{a}_i)_{i<\omega_1}$ is a subsequence of some sequence in $\mathcal{I}_n$, then $(\bar{a}_i)_{i<\omega_1} \in \mathcal{I}_n$.
4. For each $I = (\bar{a}_i)_{i<\omega_1} \in \mathcal{I}_n$ and $k < \omega_1$ the automorphism $f_k$ extends the mapping $\bar{a}_i \mapsto \bar{a}_{i+k}$ for each $i < \omega_1$.
5. For every $n < \omega$, $I \in \mathcal{I}_{n+1}$ and $k < \omega_1$, we have that $f_k(\mathcal{A}_n) \cup (f_k)^{-1}(\mathcal{A}_n) \subset \mathcal{A}_{n+1}$.

First let $\mathcal{I}_0 = \emptyset$ and $\mathcal{A}_0$ be any $\aleph_0$-saturated model of size $\lambda$ containing $E$. Assume we have defined $\mathcal{A}_m$ and $\mathcal{I}_m$ for every $m \leq n$ and chosen $f_k$ for every $k < \omega_1$ and $I \in \mathcal{I}_n$. Then for every pair of tuples $(\bar{a}_0, \bar{a}_1) \in \mathcal{A}_n$ such that it is the beginning of some strongly $E$-indiscernible sequence, choose one such sequence $I_{(\bar{a}_0, \bar{a}_1)}$ such that $|I_{(\bar{a}_0, \bar{a}_1)}| = \aleph_1$. Then let $\mathcal{I}'_{n+1} = \{I_{(\bar{a}_0, \bar{a}_1)} : (\bar{a}_0, \bar{a}_1) \in \mathcal{A}_n \text{ extends to a strongly } E\text{-indiscernible sequence}\}$. The size of $\mathcal{I}'_{n+1}$ is at most $\aleph_1 \times \{|(\bar{a}_0, \bar{a}_1) : \bar{a}_0, \bar{a}_1 \in \mathcal{A}_n|\} = \lambda$. Then define $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{J : |J| = \omega_1 \text{ and } J \text{ is a subsequence of some } I \in \mathcal{I}'_{n+1}\}$. Now $|\mathcal{I}_{n+1}| \leq |\mathcal{I}_n| + (2^{\aleph_1} \times |\mathcal{I}'_{n+1}|) = \lambda$. For every $I \in \mathcal{I}_{n+1} \setminus \mathcal{I}_n$ and $k < \omega_1$ choose an automorphism $f_k$ satisfying (iv). This is possible, since $I$ is strongly $E$-indiscernible. Finally define

$$A_{n+1} = \mathcal{A}_n \cup \bigcup \mathcal{I}_{n+1} \cup \bigcup_{I \in \mathcal{I}_{n+1}, k < \omega_1} (f_k(\mathcal{A}_n) \cup (f_k)^{-1}(\mathcal{A}_n)),$$

and let $\mathcal{A}_{n+1}$ be some $\aleph_0$-saturated model containing $A_{n+1}$ such that $|\mathcal{A}_{n+1}| = |A_{n+1}| = \lambda$. 

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We let $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$ and $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ and claim that these satisfy (1)-(3). Item (1) follows from (iii) and (2) follows from (ii). For (3), let $I = (\bar{a}_i)_{i<\omega_1} \in \mathcal{I}$ and $k < \omega_1$. It is enough to show that $f^I_k(\mathcal{A}) = \mathcal{A}$, but this follows from (v). Also $E \subseteq \mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{A}$ is $\aleph_0$-saturated, since it is an infinite union of $\aleph_0$-saturated models.

5.1. Lascar strong types and strong automorphisms. Until now we have talked about a monster model $\mathcal{M}$, which is large enough for our purposes. Here we remind ourselves what this large enough means.

**Definition 5.12.** Let $\mu$ be the cardinal related to the monster model as in Theorem 2.18. We say that a set $A$ is bounded, if $|A| < \mu$. Similarly an ordinal $\alpha$ is bounded, if $|\alpha| < \mu$.

Until now, and also from now on if not mentioned otherwise, we will assume that all sets and models under discussion are bounded.

**Definition 5.13.** We say that a set $X$ is $E$-invariant, if for every $f \in \text{Aut}(\mathcal{M}/E)$, $f(X) = X$.

Similarly a relation $R$ is $E$-invariant, if it is preserved under all $f \in \text{Aut}(\mathcal{M}/E)$, i.e. $R(a_0, ..., a_n)$ if and only if $R(f(a_0), ..., f(a_n))$.

Now we see that if a set $X$ is both bounded and $E$-invariant for some countable set $E$, then $|X| \leq \aleph_0$. Otherwise we would get from Corollary 5.9 some $a_0 \in X$, such that $(a_0)$ is the beginning of a strongly $E$-indiscernible sequence of length greater than $|X|$. Then we would have an automorphism mapping $a_0$ outside of $X$ and fixing $E$ pointwise. This contradicts the $E$-invariance of $X$.

This definition of Lascar strong type is analogous to the one in [7].

**Definition 5.14 (Lascar strong type).** We say that $\bar{a}$ and $\bar{b}$ have the same Lascar strong type over $E$, written

$$Lstp(\bar{a}/E) = Lstp(\bar{b}/E),$$

if $\ell(\bar{a}) = \ell(\bar{b})$ and $E(\bar{a}, \bar{b})$ holds for any $E$-invariant equivalence relation $E$ of $\ell(\bar{a})$-tuples with a bounded number of classes.

**Lemma 5.15.** Let $E$ be a $E$-invariant equivalence relation of $n$-tuples with a bounded number of classes. Let $(\bar{a}_i)_{i<\lambda}$, $\ell(\bar{a}_0) = n$, be strongly indiscernible over $E$. Then $E(\bar{a}_i, \bar{a}_j)$ for any $i, j < \lambda$.

**Proof.** If not, then $\neg E(\bar{a}_{i_0}, \bar{a}_{j_0})$ for some $i_0, j_0 < \lambda$. By symmetry we may choose $i_0 < j_0$. Let $\kappa$ be the number of equivalence classes of $E$. By strong indiscernibility we can extend $(\bar{a}_i)_{i<\lambda}$ to $(\bar{a}_i)_{i<\alpha}$ for any ordinal $\alpha < \kappa^+$, and thus we can extend
it to \((\bar{a}_i)_{i<\kappa^+}\). But now, by \(E\)-invariance and strong indiscernibility again, we have \(\neg E(\bar{a}_i, \bar{a}_j)\) for any \(i < j < \kappa^+\), and thus \(E\) has more than \(\kappa\) equivalence classes, a contradiction.

\[
\text{Lemma 5.16. Let } E \text{ be countable and } E \text{ an } E\text{-invariant equivalence relation for } n\text{-tuples, with a bounded number of classes. Then } E \text{ has at most countably many classes.}
\]

\[
\begin{proof}
\text{Suppose that } (\bar{a}_i)_{i<\omega_1} \text{ are } E\text{-inequivalent. By Corollary 5.9 there are } \bar{a}_{i_0}, \bar{a}_{i_1} \text{ such that } (\bar{a}_{i_0}, \bar{a}_{i_1}) \text{ is the beginning of a strongly indiscernible sequence over } E. \text{ But then we have that } E(\bar{a}_{i_0}, \bar{a}_{i_1}) \text{ by the previous lemma, a contradiction.}
\end{proof}
\]

\[
\text{Proposition 5.17. Assume that } \bar{a} \neq \bar{b} \text{ and } E \text{ is countable. Then the following are equivalent.}
\]

\[
\begin{enumerate}
\item \(\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)\).
\item \(\text{There exists } n < \omega, \bar{a}_i \text{ and strongly } E\text{-indiscernible sequences } J_i \text{ for } i \leq n \text{ such that } \bar{a}_0 = \bar{a}, \bar{a}_n = \bar{b} \text{ and } \bar{a}_i, \bar{a}_{i+1} \in J_i \text{ for } i < n.\)
\end{enumerate}
\]

\[
\begin{proof}
\text{By Lemma 5.15, (2) implies that } E(\bar{a}, \bar{b}) \text{ for every } E\text{-invariant equivalence relation } E \text{ with a bounded number of classes, and thus (1). To see that (1) implies (2), let } E(\bar{a}, \bar{b}) \text{ if the condition defined by (2) holds. The relation } E \text{ is } E\text{-invariant by Remark 5.5. We can also easily see that it is an equivalence relation, since there is no requirement about the place or order of } \bar{a}_i \text{ and } \bar{a}_{i+1} \text{ in } J_i. \text{ Then it is left to show that } E \text{ has a bounded number of classes. Assume that there would be } (\bar{b}_i)_{i<\omega_1} \text{ such that } \neg E(\bar{b}_i, \bar{b}_j) \text{ for any } i, j < \omega_1. \text{ But then by Corollary 5.9 there are } i_0 < i_1 < \omega_1 \text{ such that } (\bar{b}_{i_0}, \bar{b}_{i_1}) \text{ is the beginning of a strongly } E\text{-indiscernible sequence, a contradiction.}
\end{proof}
\]

\[
\text{Corollary 5.18. Let } E \text{ be countable. The relation } E(\bar{a}, \bar{b}) \text{ given by}
\]

\[
\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)
\]

\[
\text{is the finest } E\text{-invariant equivalence relation of } \ell(\bar{a})\text{-tuples with a bounded number of classes.}
\]

\[
\begin{proof}
\text{Denote } n = \ell(\bar{a}). \text{ Clearly } E \text{ is an } E\text{-invariant equivalence relation of } n\text{-tuples, and it is finer that any } E\text{-invariant equivalence relation with a bounded number of classes. It is left to show that it has a bounded number of classes. Assume that } (\bar{a}_i)_{i<\omega_1} \text{ realize distinct Lascar strong types over } E. \text{ But by Corollary 5.9, there are } i_0 < i_1 < \omega_1 \text{ such that } (\bar{a}_{i_0}, \bar{a}_{i_1}) \text{ is the beginning of a strongly } E\text{-indiscernible sequence. Then by the previous Proposition we have that } \text{Lstp}(\bar{a}_{i_0}/E) = \text{Lstp}(\bar{a}_{i_1}/E), \text{ a contradiction.}
\end{proof}
\]
We remark that using a a similar result to the one in 2.22, instead of Corollary 5.9, we could also prove Proposition 5.17 and Corollary 5.18 without the assumption that $E$ is countable. In this paper it is enough to study countable sets $E$.

**Definition 5.19** (Bounded closure). Let $E$ be a set. Denote $p_a(E) = \{ b \in \mathcal{M} : \text{tp}^w(b/E) = \text{tp}^w(a/E) \}$. We say that an element $a$ is in the bounded closure of $E$, written $a \in \text{bcl}(E)$, if the set $p_a(E)$ is bounded. We also say that a weak type of a tuple $\text{tp}^w(\bar{a}/E)$ is bounded, if $p_a(E)$ is bounded.

Actually we get from Lemma 5.3, that for all $E$, if $p_a(E)$ is bounded, then $|p_a(E)| \leq |E| + \aleph_0$. Otherwise we would find a sequence $(\bar{a}_i)_{i<\omega_1} \subset p_a(E)$, such that it is a Morley sequence over some $\aleph_0$-saturated model containing $E$. Then, as in the proof of Lemma 5.15, we could stretch the sequence to the length of $|p_a(E)|^+$ using extension property. Since all tuples in the Morley sequence have same weak type over $E$, this would be a contradiction. Furthermore also $|\text{bcl}(E)| \leq |E| + \aleph_0$. Otherwise there would have to be $(|E| + \aleph_0)^+$-many elements with different weak type over $E$, again a contradiction with Lemma 5.3. With similar reasoning for tuples $\bar{a}$, we get that $|\{ \bar{a} \in \mathcal{M} : \text{tp}^w(\bar{a}/E) \text{ is bounded} \}| \leq |E| + \aleph_0$.

Here a few lemmas to describe the nature of the bounded closure.

**Lemma 5.20.** The following are equivalent:

1. $\text{tp}^w(\bar{a}/E)$ is bounded.
2. $p_a(E) \subset \mathcal{A}$ for every model $\mathcal{A}$ such that $E \subset \mathcal{A}$.

**Proof.** Item (1) clearly follows from (2). We show that (2) follows from (1). Assume that $\bar{b} \in p_a(E) \setminus \mathcal{A}$ for some model such that $E \subset \mathcal{A}$. Let $\mathcal{B}$ be a model containing both $\mathcal{A}$ and $\bar{b}$. We claim that for every $i < (|E| + \aleph_0)^+$ there is a model $\mathcal{B}_i$, and an isomorphism $f_i : \mathcal{B} \to \mathcal{B}_i$, $f_i \upharpoonright \mathcal{A} = \text{id}_\mathcal{A}$, such that when $i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \mathcal{A}$. Let $\mathcal{B}_0 = \mathcal{B}$. Assume that we have defined $B_j$ for $j < i$, and let $\mathcal{C}$ be a model containing all of them. Then let $\mathcal{B}_i'$, not necessarily a substructure of $\mathcal{M}$, be such that $\mathcal{B}_i' \cap \mathcal{C} = \mathcal{A}$ and $f_i' : \mathcal{B} \to \mathcal{B}_i'$ an isomorphism such that $f_i' \upharpoonright \mathcal{A} = \text{id}_\mathcal{A}$. By disjoint amalgamation there is $\mathcal{D} \in \mathcal{K}$ and $g : \mathcal{C} \cup \mathcal{B}_i' \to \mathcal{D}$ such that $g \upharpoonright \mathcal{C}$ and $g \upharpoonright \mathcal{B}_i'$ are $\mathcal{K}$-embeddings and $g(\mathcal{C}) \cap g(\mathcal{B}_i') = g(\mathcal{A})$. We may assume that $|\mathcal{D}| = |\mathcal{C}|$. By universality of $\mathcal{M}$ we have a $\mathcal{K}$-embedding $h : \mathcal{D} \to \mathcal{M}$, and $(h \circ g)^{-1} : h(g(\mathcal{A})) \to \mathcal{A}$ extends to an automorphism of $\mathcal{M}$, say $F$. Finally we can take $B_i = F(h(g(\mathcal{B}_i'))) = f_i = F \circ h \circ g \circ f_i'$. This proves the claim. When we denote $\bar{b}_i = f_i(\bar{b})$ for $i < (|E| + \aleph_0)^+$, we get that when $i \neq j$, $\bar{b}_i \neq \bar{b}_j$, and $\text{tp}^g(\bar{b}_i/\mathcal{A}) = \text{tp}^g(\bar{b}_j/\mathcal{A})$ and thus also $\text{tp}^w(\bar{b}_i/E) = \text{tp}^w(\bar{a}/E)$. Hence $\text{tp}^w(\bar{a}/E)$ is not bounded.

**Lemma 5.21.** (1) If $E_1 \subset E_2$, we have that $E_1 \subset \text{bcl}(E_1) \subset \text{bcl}(E_2)$. 
(2) If $E$ is finite, we have that $\text{bcl}(E) = \text{bcl}(\text{bcl}(E))$.

(3) If $E$ is finite, we have that a tuple $\bar{a} \in \text{bcl}(E)$ if and only if $\text{tp}^w(\bar{a}/E)$ is bounded.

Proof. Item (1) is clear, since if $a \in E_1 \subset E_2$, $p_a(E_1) = \{a\}$ and $p_b(E_2) \subset p_b(E_1)$ for all $b$. By 1, $\text{bcl}(E) \subset \text{bcl}(\text{bcl}(E))$ for all $E$. Assume that $a \notin \text{bcl}(E)$ and $E$ finite. Let $\mathcal{A}$ be some model containing $E$. Since $p_a(E)$ is not bounded, there is $b \in p_a(E)$ such that $b \notin \mathcal{A}$. Since $E$ is finite, there is $f \in \text{Aut}(\mathcal{M}/E)$ such that $f(b) = a$. Now $f(\mathcal{A})$ is a model containing $E$ and $a \notin f(\mathcal{A})$. By Lemma 5.20, $\text{bcl}(E) \subset \mathcal{A}$, and furthermore $\text{bcl}(\text{bcl}(E)) \subset f(\mathcal{A})$. Thus $a \notin \text{bcl}(\text{bcl}(E))$, and this proves (2).

For (3), let $E$ be finite and $\bar{a} = (a_0, ..., a_{n-1})$. If $p_a(E)$ is bounded for each $0 \leq i < n$, also $p_{\bar{a}}(E)$ must be bounded. Then assume that $\bar{a} \notin \text{bcl}(E)$, and thus there is $i$ such that $p_{\bar{a}}(E)$ is not bounded. Let $(\bar{b}_j)_{j<\omega_1}$ be distinct such that $\text{tp}^w(\bar{b}_j/E) = \text{tp}^w(\bar{a}_j/E)$ for each $j < \omega_1$. But now for each $j < \omega_1$ there is $f_j \in \text{Aut}(\mathcal{M}/E)$ such that $f_j(\bar{a}_i) = \bar{b}_j$. Then $(f_j(\bar{a}))_{j<\omega_1}$ are distinct tuples in $p_{\bar{a}}(E)$, and thus $p_{\bar{a}}(E)$ is not bounded.

Lemma 5.22. Let $E$ be a set. Assume that $\text{tp}^w(\bar{a}/E)$ is bounded and $\bar{b}, \bar{c}$ are its realizations. Then $\text{Lstp}(\bar{b}/E) = \text{Lstp}(\bar{c}/E)$ if and only if $\bar{b} = \bar{c}$.

Proof. We can define an equivalence relation $E$ such that $E(\bar{b}, \bar{c})$, if $\bar{b}, \bar{c} \notin p_{\bar{a}}(E)$, or $\bar{b} = \bar{c}$. Thus $E$ is $E$-invariant and has $|p_{\bar{a}}(E)| + 1$ classes. The result follows from the definition of Lascar strong type.

Lemma 5.23. If $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$, then $\text{tp}^w(\bar{a}/\text{bcl}(E)) = \text{tp}^w(\bar{b}/\text{bcl}(E))$.

Proof. Define $E(\bar{a}, \bar{b})$ if $\text{tp}^w(\bar{a}/\text{bcl}(E)) = \text{tp}^w(\bar{b}/\text{bcl}(E))$. This is a $E$-invariant equivalence relation, for if $f \in \text{Aut}(\mathcal{M}/E)$, then $f$ fixes $\text{bcl}(E)$ setwise. Also by Lemma 5.3, there are at most $|\text{bcl}(E)| + \aleph_0$ different weak types over the set $\text{bcl}(E)$, thus $E$ has only a bounded number of classes. Hence if $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$, then $E(\bar{a}, \bar{b})$.

Another way to define the concept of an algebraic closure, so called essential closure, is studied in [8].

Definition 5.24 (Essential closure). Denote $p^\emptyset_a(E) = \{\bar{b} \in \mathcal{M} : \text{tp}^g(\bar{b}/E) = \text{tp}^g(\bar{a}/E)\}$. We say that an element $a$ is in the essential closure of a set $E$, written $a \in \text{ecl}(E)$, if the set $p^\emptyset_a(E)$ is bounded.

We see that for any $E$, $\text{bcl}(E) \subset \text{ecl}(E)$. Also essential closure is a so called closure operator, i.e. it satisfies that if $E_1 \subset E_2$, then $E_1 \subset \text{ecl}(E_1) \subset \text{ecl}(E_2)$, and $\text{ecl}(E) = \text{ecl}(\text{ecl}(E))$ for any $E$. This can be seen with a similar proof to the proof
of Lemmas 5.20 and 5.21. Also for every set $E$, $tp^g(\bar{a}/E)$ is bounded if and only if $\bar{a} \in ecl(E)$. Under simplicity\footnote{See section 6.} the bounded closure is also a closure operator.

Equality of weak types and Galois types over countable models gives us another equivalence: $\bar{a}$ and $\bar{b}$ having the same Lascar strong type over a countable $E$ is equivalent to the existence of a Strong automorphism fixing $E$ and mapping $\bar{a}$ to $\bar{b}$. Again the restriction on the size of $E$ would be loosened by tameness.

**Definition 5.25** (Strong automorphism). We say that $f \in \text{Aut}(\mathcal{M}/E)$ is a strong automorphism over $E$ if $\text{Lstp}(\bar{a}/E) = \text{Lstp}(f(\bar{a})/E)$ for each tuple $\bar{a}$.

Denote by $Saut(\mathcal{M}/E)$ the group of strong automorphisms over $E$.

**Remark 5.26.** The group $Saut(\mathcal{M}/E)$ is a normal subgroup of $\text{Aut}(\mathcal{M}/E)$. That is, if $f \in Saut(\mathcal{M}/E)$ and $g \in \text{Aut}(\mathcal{M}/E)$, then also $(g \circ f \circ g^{-1}) \in Saut(\mathcal{M}/E)$.

**Proof.** Let $f \in Saut(\mathcal{M}/E)$, $g \in \text{Aut}(\mathcal{M}/E)$ and $\bar{a}$ be a tuple. We have that $\text{Lstp}(g^{-1}(\bar{a})) = \text{Lstp}(f(g^{-1}(\bar{a})))$. Since equality of Lascar strong types over $E$ is $E$-invariant, we get that $\text{Lstp}(g^{-1}(\bar{a})) = \text{Lstp}(g(f(\bar{a})))$ and thus $\text{Lstp}(\bar{a}) = \text{Lstp}(g(\bar{a}))$.

**Proposition 5.27.** Let $E$ be countable. The following are equivalent:

1. $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$.
2. There exists $f \in Saut(\mathcal{M}/E)$ such that $f(\bar{a}) = \bar{b}$.

**Proof.** By definition, (2) implies (1). We prove that (1) implies (2). Let $E(\bar{a}, \bar{b})$ hold if there is $f \in Saut(\mathcal{M}/E)$ such that $f(\bar{a}) = \bar{b}$. Now by Remark 5.26 the relation $E$ is $E$-invariant. Hence it is enough to show that it has a bounded number of classes. By Corollary 5.18, equivalence of Lascar strong type over $E$ is an equivalence relation with a bounded number of classes, and thus has at most countably many classes. We can choose an $\aleph_0$-saturated countable model $\mathcal{A}$ such that it contains a realization for each Lascar strong type over $E$. Assume that $\{\bar{a}_i : i < \omega_1\}$ are $E$-inequivalent. By $\aleph_0$-stability there are $i < j < \omega_1$ such that $tp^w(\bar{a}_i/\mathcal{A}) = tp^w(\bar{a}_j/\mathcal{A})$ and since by Theorem 3.12 also $tp^g(\bar{a}_i/\mathcal{A}) = tp^g(\bar{a}_j/\mathcal{A})$. Let $f \in \text{Aut}(\mathcal{M}/\mathcal{A})$ be such that $f(\bar{a}_i) = \bar{a}_j$. We claim that actually $f \in Saut(\mathcal{M}/\mathcal{E})$. Let $\bar{a}$ be a tuple. Now there is some $\bar{a}' \in \mathcal{A}$ such that $\text{Lstp}(\bar{a}'/E) = \text{Lstp}(\bar{a}/E)$. But then $\text{Lstp}(f(\bar{a}')/E) = \text{Lstp}(f(\bar{a})/E)$ by $E$-invariance. Since $f(\bar{a}') = \bar{a}'$, we have that $\text{Lstp}(f(\bar{a})/E) = \text{Lstp}(\bar{a}/E)$.

It follows that if $E$ is countable and $\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)$, then for each $\bar{c}$ there is $\bar{d}$ such that $\text{Lstp}(\bar{a} \sim \bar{c}/E) = \text{Lstp}(\bar{b} \sim \bar{d}/E)$.

The letter $a$ in the following definition comes from $F_{\aleph_0}^a$ in [14].
Definition 5.28. We say that a model $\mathcal{A}$ is $\mathfrak{a}$-saturated, if for every finite $E \subset \mathcal{A}$ and $\bar{a}$, there is $b \in \mathcal{A}$ such that $\operatorname{Lstp}(\bar{b}/E) = \operatorname{Lstp}(\bar{a}/E)$.

Proposition 5.29. Every $\aleph_0$-saturated model is also $\mathfrak{a}$-saturated.

Proof. It is enough to prove the claim for every countable $\aleph_0$-saturated model $\mathcal{A}$. Let $\mathcal{A}$ be a countable $\aleph_0$-saturated model and $E \subset \mathcal{A}$ finite. Since there are only countably many different Lascar strong types over $E$, there is an $\aleph_0$-saturated countable model $\mathcal{B}$ such that $E \subset \mathcal{B}$ and every Lascar strong type over $E$ is represented in $\mathcal{B}$. But since both models are countable and $\aleph_0$-saturated, there is $f \in \operatorname{Aut}(\mathcal{M}/E)$ such that $F(\mathcal{A}) = \mathcal{B}$. Now if there would be a tuple $\bar{a}$ such that $\operatorname{Lstp}(\bar{a}/E) \neq \operatorname{Lstp}(\bar{b}/E)$ for every $\bar{b} \in \mathcal{A}$, then we would have that $\operatorname{Lstp}(f^{-1}(\bar{a})/E) \neq \operatorname{Lstp}(\bar{b}/E)$ for every $\bar{b} \in \mathcal{B}$, a contradiction. $\square$

5.2. Lascar splitting and independence. Also the notions of Lascar splitting and independence are analogous to the ones in [7].

Definition 5.30. We say that $\operatorname{tp}^w(\bar{a}/A)$ Lascar-splits over finite $E$ if there is a strongly $E$-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $a_0, a_1 \in A$ and $\operatorname{tp}^0(\bar{a}_0/E \cup \{\bar{a}\}) \neq \operatorname{tp}^0(\bar{a}_1/E \cup \{\bar{a}\})$.

If $\operatorname{tp}^w(\bar{a}/A)$ does not split over finite $E$, then $\operatorname{tp}^w(\bar{a}/A)$ does not Lascar split over $E$. Thus we get from Theorem 3.16 that for every model $\mathcal{A}$ and tuple $\bar{a}$ there is finite $E \subset \mathcal{A}$ such that $\operatorname{tp}^w(\bar{a}/\mathcal{A})$ does not Lascar-split over $E$.

Also clearly if $\operatorname{tp}^w(\bar{a}/A) = \operatorname{tp}^w(\bar{b}/A)$, then $\operatorname{tp}^w(\bar{a}/A)$ Lascar-splits over a finite $E \subset A$ if and only if $\operatorname{tp}^w(\bar{b}/A)$ does.

Proposition 5.31. Let $\mathcal{A}$ be an $\aleph_0$-saturated model and $E \subset \mathcal{A}$ finite. Then $\operatorname{tp}^w(\bar{a}/\mathcal{A})$ Lascar-splits over $E$ if and only if there are $\bar{c}, \bar{d} \in \mathcal{A}$ such that $\operatorname{Lstp}(\bar{c}/E) = \operatorname{Lstp}(\bar{d}/E)$ but $\operatorname{tp}^0(\bar{c}/E \cup \{\bar{a}\}) \neq \operatorname{tp}^0(\bar{d}/E \cup \{\bar{a}\})$.

Proof. If $(\bar{a}_i)_{i<\omega}$ is strongly indiscernible, then $\operatorname{Lstp}(\bar{a}_0/E) = \operatorname{Lstp}(\bar{a}_1/E)$ by Lemma 5.15. Thus if $\operatorname{tp}^w(\bar{a}/\mathcal{A})$ Lascar-splits over $E$, such $\bar{c}$ and $\bar{d}$ exist, namely $\bar{a}_0$ and $\bar{a}_1$ from the definition of Lascar-splitting.

To prove the other direction, we assume that $\operatorname{tp}^w(\bar{a}/\mathcal{A})$ does not Lascar-split over $E$ and $\operatorname{Lstp}(\bar{c}/E) = \operatorname{Lstp}(\bar{d}/E)$ for some $\bar{c}, \bar{d} \in \mathcal{A}$. From Proposition 5.17 we get $n < \omega$, $a_i$ for $i \leq n$ and strongly indiscernible $J_i$ for $i < n$ such that $\bar{c} = \bar{a}_0$, $\bar{d} = \bar{a}_n$ and $\bar{a}_i, \bar{a}_{i+1} \in J_i$ for $i < n$. By $\aleph_0$-saturation of $\mathcal{A}$, we have $f \in \operatorname{Aut}(\mathcal{M}/E \cup \{\bar{c}, \bar{d}\})$ such that $f(\bar{a}_i) \in \mathcal{A}$ for each $i \leq n$. Then $f(J_i)$ is strongly indiscernible for each $i < n$. By taking a suitable subsequence of $f(J_i)$ we get that there are strongly indiscernible sequences $I_i$ such that $\bar{a}_i$ and $\bar{a}_{i+1}$ are the two first elements of the sequence (one or other being the first) and belong
in \( \mathcal{A} \) for each \( i < n \). Then since \( \text{tp}^w(\bar{a}/\mathcal{A}) \) does not Lascar-split over \( E \), we have that \( \text{tp}^g(\bar{a}_i/E \cup \{\bar{a}\}) = \text{tp}^g(\bar{a}_{i+1}/E \cup \{\bar{a}\}) \) for each \( i < n \) and thus also \( \text{tp}^g(\bar{b}/E \cup \{\bar{a}\}) = \text{tp}^g(\bar{c}/E \cup \{\bar{a}\}) \). \( \square \)

Now we see that if \( \text{tp}^w(\bar{a}/\mathcal{A}) \) Lascar-splits over finite \( E \subset \mathcal{A} \) and \( E' \subset E \), then \( \text{tp}^w(\bar{a}/\mathcal{A}) \) Lascar-splits over \( E' \).

**Lemma 5.32.** Assume that \( \mathcal{A} \) and \( \mathcal{B} \) are countable \( \mathbb{N}_0 \)-saturated models and \( E \subset \mathcal{A} \cap \mathcal{B} \) a finite set. Then there is \( f \in \text{Aut}(\mathcal{M}/E) \) such that \( f(\mathcal{A}) = \mathcal{B} \) and for every finite \( \bar{a} \in \mathcal{A} \) there is \( g \in \text{Saut}(\mathcal{M}/E) \) such that \( g(\bar{a}) = f(\bar{a}) \).

**Proof.** By Proposition 5.29, both models \( \mathcal{A} \) and \( \mathcal{B} \) are also a-saturated. The proof of this lemma is a similar back-and-forth construction as the proof of Lemma 3.8. We only take the functions \( f_n \) in the construction to be strong automorphisms by a-saturation. \( \square \)

**Proposition 5.33.** Assume that \( \mathcal{A} \preceq \mathcal{B} \) and both are \( \mathbb{N}_0 \)-saturated models. If \( \text{tp}^w(\bar{c}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{B}) \), \( \text{tp}^w(\bar{b}/\mathcal{B}) \) does not Lascar-split over some finite \( E \subset \mathcal{A} \) and \( \text{tp}^w(\bar{c}/\mathcal{B}) \) does not Lascar-split over some finite \( E' \subset \mathcal{A} \), then \( \text{tp}^w(\bar{c}/\mathcal{B}) = \text{tp}^w(\bar{b}/\mathcal{B}) \).

**Proof.** By Proposition 5.29 the model \( \mathcal{A} \) is also a-saturated. The proof of this Proposition is the same as the proof of stationarity in Theorem 3.17 if we use Proposition 5.31 and a-saturation instead of \( \mathbb{N}_0 \)-saturation. \( \square \)

The extension property for Lascar-splitting follows from this property for splitting.

**Proposition 5.34.** Assume that \( \mathcal{A} \) is an \( \mathbb{N}_0 \)-saturated model, \( \mathcal{A} \subset \mathcal{B} \) and \( \text{tp}^w(\bar{a}/\mathcal{A}) \) does not Lascar-split over a finite \( E \subset \mathcal{A} \). There exists \( \bar{b} \) such that \( \text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \) and \( \text{tp}^w(\bar{b}/\mathcal{B}) \) does not Lascar-split over \( E \).

**Proof.** We may assume that \( \mathcal{B} = \mathcal{B} \) is an \( \mathbb{N}_0 \)-saturated model. First we prove the claim for countable \( \mathcal{A} \) and \( \mathcal{B} \). Let \( f \in \text{Aut}(\mathcal{M}/E) \) be the automorphism from Lemma 5.32, mapping \( \mathcal{A} \) onto \( \mathcal{B} \). Then \( \text{tp}^w(f(\bar{a})/\mathcal{B}) \) does not Lascar split over \( E \). When \( C \subset \mathcal{A} \) is an arbitrary finite set, we get by Lemma 5.32 that \( \text{Lstp}(C/E) = \text{Lstp}(f(C)/E) \). Since \( C \cup f(C) \subset \mathcal{B} \) and \( \text{tp}^w(f(\bar{a})/\mathcal{B}) \) does not Lascar split over \( E \), there is \( h \in \text{Aut}(\mathcal{M}/E \cup \{f(\bar{a})\}) \) such that \( h \upharpoonright C = f \upharpoonright C \). Then \( f^{-1} \circ h \) witnesses that \( \text{tp}^g(f(\bar{a})/C) = \text{tp}^g(\bar{a}/C) \). Since \( C \) was arbitrary, we get that \( \text{tp}^w(f(\bar{a})/\mathcal{A}) = \text{tp}^w(\bar{a}/\mathcal{A}) \). We can take \( \bar{b} = f(\bar{a}) \).

Then let \( \mathcal{A} \) and \( \mathcal{B} \) be of arbitrary size. By extension property for splitting there exists \( \bar{b} \) such that \( \text{tp}^w(\bar{b}/\mathcal{A}) = \text{tp}^w(\bar{a}/\mathcal{A}) \) and \( \text{tp}^w(\bar{b}/\mathcal{B}) \) does not split over some
finite \( E' \subset \mathcal{A} \). Thus \( \text{tp}^w(\bar{b}/\mathcal{B}) \) does not Lascar-split over \( E' \) either. We should show that \( \text{tp}^w(\bar{b}/\mathcal{B}) \) does not Lascar-split over \( E \).

Assume that \((\bar{a}_i)_{i<\omega}\) is strongly \( E \)-indiscernible such that \( \bar{a}_0, \bar{a}_1 \in B \). Let then \( \mathcal{A}_0 \preceq^E \mathcal{A} \) be a countable and \( \aleph_0 \)-saturated model such that \( E \cup E' \subset \mathcal{A}_0 \). Let also \( \mathcal{B}_0 \preceq^E \mathcal{B} \) be countable and \( \aleph_0 \)-saturated containing \( \mathcal{A}_0 \cup \{\bar{a}_0, \bar{a}_1\} \). By the countable case there exists \( \bar{b}' \) such that \( \text{tp}^w(\bar{b}'/\mathcal{A}_0) = \text{tp}^w(\bar{a}/\mathcal{A}_0) \) and \( \text{tp}^w(\bar{b}'/\mathcal{B}_0) \) does not Lascar-split over \( E \). Thus we have that \( \text{tp}^q(\bar{a}_0/E \cup \{\bar{b}'\}) = \text{tp}^q(\bar{a}_1/E \cup \{\bar{b}'\}) \).

On the other hand, we have that \( \text{tp}^w(\bar{b}/\mathcal{B}_0) \) does not Lascar-split over \( E' \subset \mathcal{A}_0 \), \( \text{tp}^w(\bar{b}'/\mathcal{B}_0) \) does not Lascar-split over \( E \subset \mathcal{A}_0 \) and \( \text{tp}^w(\bar{b}'/\mathcal{A}_0) = \text{tp}^w(\bar{b}/\mathcal{A}_0) \). Then we get from Proposition 5.33 that \( \text{tp}^w(\bar{b}'/\mathcal{A}_0 \cup \{\bar{a}_0, \bar{a}_1\}) = \text{tp}^w(\bar{b}/\mathcal{A}_0 \cup \{\bar{a}_0, \bar{a}_1\}) \). Hence also \( \text{tp}^q(\bar{a}_0/E \cup \{\bar{b}\}) = \text{tp}^q(\bar{a}_1/E \cup \{\bar{b}\}) \).

**Lemma 5.35.** If \( E \) is finite and \( \text{tp}^w(\bar{a}/E) \) is bounded, then \( \text{tp}^w(\bar{a}/B) \) does not Lascar-split over \( E \) for any \( B \).

**Proof.** Assume that \( \text{tp}^w(\bar{a}/B) \) does split over \( E \). Let \((\bar{b}_i)_{i<\omega_1}\) be strongly \( E \)-indiscernible such that \( \text{tp}^q(\bar{b}_0/E \cup \{\bar{a}\}) \neq \text{tp}^q(\bar{b}_1/E \cup \{\bar{a}\}) \). There has to be either uncountably many \( i \) such that \( \text{tp}^q(\bar{b}_i/E \cup \{\bar{a}\}) \neq \text{tp}^q(\bar{b}_0/E \cup \{\bar{a}\}) \) or uncountably many \( i \) such that \( \text{tp}^q(\bar{b}_i/E \cup \{\bar{a}\}) \neq \text{tp}^q(\bar{b}_1/E \cup \{\bar{a}\}) \). Thus we may assume that \( \text{tp}^w(\bar{b}_i/E \cup \{\bar{a}\}) \neq \text{tp}^w(\bar{b}_0/E \cup \{\bar{a}\}) \) for each \( i < \omega_1 \). By strong \( E \)-indiscernibility, for each \( i < \omega_1 \), there is \( f_i \in \text{Aut}((\mathcal{M}/E) \) such that \( f_i(\bar{b}_0) = \bar{b}_{i+k} \) for each \( k < \omega_1 \). Now if \( i < j \) we have that \( f_i(\bar{a}) \neq f_j(\bar{a}) \). Otherwise we would have that \( (f_i^{-1} \circ f_j)(\bar{a}) = \bar{a} \) and \( (f_i^{-1} \circ f_j)(\bar{b}_0) = \bar{b}_k \) for \( k > 0 \). Now \((f_i(\bar{a}))_{i<\omega_1}\) are different realizations of \( \text{tp}^w(\bar{a}/E) \), which contradicts the assumption that \( \text{tp}^w(\bar{a}/E) \) is bounded.

As in [9] and [7], the notion of independence has a built-in extension property.

**Definition 5.36** (Independence). We say that \( \bar{a} \) is independent of \( B \) over \( C \), write \( \bar{a} \downarrow_C B \), if there is finite \( E \subset C \) such that for all \( D \) containing \( C \cup B \) there is \( \bar{b} \) such that \( \text{tp}^w(\bar{b}/B \cup C) = \text{tp}^w(\bar{a}/B \cup C) \) and \( \text{tp}^w(\bar{b}/D) \) does not Lascar-split over \( E \). We then write \( A \downarrow_C B \), if \( \bar{a} \downarrow_C B \) for every finite tuple \( \bar{a} \in A \).

Now we show some properties of the independence notion. Similar propositions can be found in [7], and also the proofs are quite similar.

**Proposition 5.37.**

1. **Invariance:** The notion \( \downarrow \) is invariant under automorphisms of \( \mathcal{M} \). Furthermore, if \( \text{tp}^w(\bar{a}/B \cup C) = \text{tp}^w(\bar{b}/B \cup C) \) and \( \bar{a} \downarrow_C B \), then \( \bar{b} \downarrow_C B \).
(2) **Restricted local character:** If \( \bar{a} \downarrow_C B \), then there exists finite \( C' \subset C \) such that \( \bar{a} \downarrow_{C'} B \).

(3) **Monotonicity:** Assume that \( C \subset B \subset D \). If \( \bar{a} \downarrow_C D \), then \( \bar{a} \downarrow_C B \) and \( \bar{a} \downarrow_B D \).

(4) **Extension:** Let \( C \subset B \). If \( \bar{a} \downarrow_C B \) and \( B \) includes \( B \), then there is \( \bar{b} \) such that \( \text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B) \) and \( \bar{b} \downarrow_C D \).

**Proof.** Items (1), (2) and (3) are clear by the definition. We prove (4). By monotonicity, we may assume that \( D = \emptyset \) is an \( \aleph_0 \)-saturated model. Since \( \bar{a} \downarrow_C B \), there is finite \( E \subset C \) and \( \bar{b} \) such that \( \text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B) \) and \( \text{tp}^w(\bar{b}/D) \) does not Lascar-split over \( E \). By Proposition 5.34 we have that \( \bar{b} \downarrow_C D \).

We remark that when \( \bar{a} \downarrow_C B \) and \( E \subset C \) is the finite set witnessing this, we have that \( \text{tp}^w(\bar{a}/C \cup B) \) does not Lascar-split over \( E \).

**Lemma 5.38.** Let \( E \) be finite.

(1) If \( \text{tp}^w(\bar{a}/E) \) is bounded, then \( \bar{a} \downarrow_E B \) for any \( B \).

(2) If \( \text{tp}^w(\bar{a}/E) \) is not bounded, then \( \bar{a} \not\downarrow_E \bar{a} \).

(3) If \( \text{tp}^w(\bar{a}/B) \) is bounded but \( \text{tp}^w(\bar{a}/E) \) is not, then \( \bar{a} \not\downarrow_E \bar{a} \).

**Proof.** Item (1) follows from Lemma 5.35. Assume that \( \text{tp}^w(\bar{a}/E) \) is not bounded, i.e. there is an uncountable set of tuples \( \bar{b} \), such that \( \text{tp}^w(\bar{b}/E) = \text{tp}^w(\bar{a}/E) \).

By Corollary 5.9, there is a strongly \( E \)-indiscernible sequence \( (\bar{a}_i)_{i<\omega} \) such that \( \text{tp}^w(\bar{a}_0/E) = \text{tp}^w(\bar{a}/E) \) and hence \( \text{tp}^w(\bar{a}_i/E) = \text{tp}^w(\bar{a}/E) \) for each \( i < \omega \). Furthermore, since we have \( f \in \text{Aut}(M/E) \) mapping \( \bar{a}_0 \) to \( \bar{a} \), we may assume that \( \bar{a}_0 = \bar{a} \). Assume, for a contradiction, that \( \bar{a} \downarrow_E \bar{a} \). Then let \( \bar{a}' \) be such that \( \text{tp}^w(\bar{a}'/E \cup \{\bar{a}\}) = \text{tp}^w(\bar{a}/E \cup \{\bar{a}\}) \) and \( \text{tp}^w(\bar{a}'/E \cup \{\bar{a}_i : i < \omega\}) \) does not Lascar-split over some \( E' \subset E \) and thus does not Lascar-split over \( E \). But now we must have that \( \bar{a}' = \bar{a} \) and this is a contradiction, since \( \text{tp}^w(\bar{a}_0/E \cup \{\bar{a}\}) \neq \text{tp}^w(\bar{a}_1/E \cup \{\bar{a}\}) \) and thus \( \text{tp}^w(\bar{a}/E \cup \{\bar{a}_i : i < \omega\}) \) does Lascar-split over \( E \). This proves (2). Then assume that \( \text{tp}^w(\bar{a}/B) \) is bounded, \( \text{tp}^w(\bar{a}/E) \) is not and \( \bar{a} \downarrow_E B \). Let \( \bar{a}' \) be such that \( \text{tp}^w(\bar{a}'/E \cup B) = \text{tp}^w(\bar{a}/E \cup B) \) and \( \bar{a}' \downarrow_E \{\bar{b} : \text{tp}^w(\bar{b}/B) \text{ is bounded}\} \). But then also \( \text{tp}^w(\bar{a}'/B) \) is bounded, and by monotonicity \( \bar{a}' \downarrow E \bar{a}' \). We have \( f \in \text{Aut}(M/E) \) such that \( f(\bar{a}') = \bar{a} \), and thus by invariance, \( \bar{a} \downarrow_E \bar{a} \). This is a contradiction with (2). We have now proven (3).

**Proposition 5.39.** Let \( \mathcal{A} \) be an \( \aleph_0 \)-saturated model.

(1) **Stationarity over \( \aleph_0 \)-saturated models:** If \( \bar{a} \downarrow_{\mathcal{A}} B \), \( \bar{b} \downarrow_{\mathcal{A}} B \) and \( \text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \), then \( \text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{b}/B) \).

(2) **Equivalence of non-Lascar-splitting and non-splitting over \( \aleph_0 \)-saturated models:** We have that \( \bar{a} \downarrow_{\mathcal{A}} B \) if and only if \( \bar{a} \downarrow^*_{\mathcal{A}} B \).
(3) **Finite character over \( \aleph_0 \)-saturated models**: If \( \bar{a} \not\ind B \), then there is a finite \( \bar{b} \in B \) such that \( \bar{a} \not\ind \bar{b} \).

(4) **Symmetry over \( \aleph_0 \)-saturated models**: If \( A' \ind B \), then \( B \ind A' \).

(5) **Transitivity when the middle set is an \( \aleph_0 \)-saturated model**: Assume that \( C \subset A \subset D \). If \( \bar{a} \ind C \) and \( \bar{a} \ind \bar{a} \ind D \), then \( \bar{a} \ind C \ind D \).

**Proof.** First we prove item (1). Let \( \mathcal{B} \) be an \( \aleph_0 \)-saturated model containing \( \mathcal{A} \cup B \). By Extension we find \( \bar{a}' \) and \( \bar{b}' \) realizing \( \tpw(\bar{a}/\mathcal{A}) \) and \( \tpw(\bar{b}/\mathcal{A}) \) respectively such that \( \bar{a}' \ind \mathcal{B} \) and \( \bar{b}' \ind \mathcal{B} \). Then item (1) follows from Proposition 5.33.

For right to left of (2), assume that \( \tpw(\bar{a}/B) \) does not split over finite \( E \subset \mathcal{A} \). For any \( D \) containing \( B \) we get from extension property for non-splitting some \( \bar{a}' \) such that \( \tpw(\bar{a}'/\mathcal{A}) = \tpw(\bar{a}/\mathcal{A}) \) and \( \tpw(\bar{a}'/D) \) does not split over \( E \), and thus does not Lascar-split over \( E \) either. Furthermore we get from stationarity for non-splitting that \( \tpw(\bar{a}'/\mathcal{A} \cup B) = \tpw(\bar{a}/\mathcal{A} \cup B) \). This shows \( \bar{a} \ind B \). Then for the right direction assume that \( \bar{a} \ind B \), and let finite \( E_1 \subset \mathcal{A} \) be as in the definition of independence. Let \( \mathcal{B} \) be \( \aleph_0 \)-saturated containing \( \mathcal{A} \cup B \). By definition there is \( \bar{a}^* \) realizing \( \tpw(\bar{a}/\mathcal{A} \cup B) \) such that \( \tpw(\bar{a}^*/\mathcal{B}) \) does not Lascar-split over \( E_1 \). Also from extension property for non-splitting we get some \( \bar{a}' \) and finite \( E_2 \subset \mathcal{A} \) such that \( \tpw(\bar{a}'/\mathcal{A}) = \tpw(\bar{a}^*/\mathcal{A}) \) and \( \tpw(\bar{a}'/\mathcal{B}) \) does not split, and hence neither Lascar-split, over \( E_2 \). Then from Proposition 5.33 we get that \( \tpw(\bar{a}'/\mathcal{B}) = \tpw(\bar{a}^*/\mathcal{B}) \), and hence \( \tpw(\bar{a}'/\mathcal{A} \cup B) = \tpw(\bar{a}/\mathcal{A} \cup B) \). Thus \( \tpw(\bar{a}/\mathcal{A} \cup B) \) does not split over \( E_2 \) and we have that \( \bar{a} \ind B \).

Item (3) follows from (2). For (4), assume that \( A' \ind B \) and \( B \ind A' \). Thus there is some \( \bar{b} \in B \) such that \( \bar{b} \ind A' \). From (3) we get some finite \( \bar{a} \in A' \) such that \( \bar{b} \ind \bar{a} \). Then from (2) and symmetry for non-splitting we get that \( \bar{a} \ind \bar{b} \), and by monotonicity that \( \bar{a} \ind B \), a contradiction.

Then assume that \( C \subset \mathcal{A} \subset D \), \( \bar{a} \ind C \) and \( \bar{a} \ind D \). By Proposition 5.37(4) there is \( \bar{a}' \) such that \( \tpw(\bar{a}'/\mathcal{A}) = \tpw(\bar{a}/\mathcal{A}) \) and \( \bar{a}' \ind D \). But then also \( \bar{a}' \ind D \) by Proposition 5.37(3), and thus \( \tpw(\bar{a}'/D) = \tpw(\bar{a}/D) \) by (1). Since now \( \bar{a} \ind D \), we have shown (5). \( \square \)

**Proposition 5.40** (Finite Pairs Lemma). Let \( B \) be finite and \( A \subset B \). Assume that \( \bar{a} \ind A \) and \( \bar{b} \ind (A \cup \{\bar{a}\}) \). Then \( \bar{a} \ind \bar{b} \ind A \).

**Proof.** Assume, for a contradiction, that \( \bar{a} \ind \bar{b} \ind A \). Especially, the finite set \( A \) does not witness that \( \bar{a} \ind \bar{b} \ind A \). Hence, there is \( D \) containing \( B \) such that whenever \( \tpw((\bar{a}')^*(\bar{b}'))/B) = \tpw(\bar{a} \ind \bar{b} / B) \), then \( \tpw((\bar{a}')^*(\bar{b}'))/D) \) Lascar-splits over \( A \). We may increase the set \( D \) if necessary, and assume that it has the following property:
For every finite \( A \subset D \) and tuples \( \bar{a}_0, \bar{a}_1 \in D \) such that they are the beginning of a strongly \( A \)-indiscernible sequence \( (\bar{a}_i)_{i<\omega_1} \), there is one such sequence in \( D \).

By definition there is \( \bar{a}' \) such that \( \text{tp}^u(\bar{a}'/B) = \text{tp}^u(\bar{a}/B) \) and \( \text{tp}^u(\bar{a}'/D) \) does not Lascar-split over \( A \). Since \( B \) is finite, we have \( f \in \text{Aut}(\mathfrak{M}/B) \) such that \( f(\bar{a}) = \bar{a}' \). Now \( \text{tp}^u((\bar{a}')^\frown f(\bar{b})/B) = \text{tp}^u(\bar{a}^\frown \bar{b}/B) \), and thus
\[
f(\bar{b}) \downarrow_{(A\cup\{\bar{a}'\})} B \cup \{\bar{a}'\}.
\]
Again by definition there is \( \bar{b}' \) such that \( \text{tp}^u(\bar{b}'/B \cup \{\bar{a}'\}) = \text{tp}^u(f(\bar{b})/B \cup \{\bar{a}'\}) \) and \( \text{tp}^u(\bar{b}'/D \cup \{\bar{a}'\}) \) does not Lascar-split over \( (A \cup \{\bar{a}'\}) \). Hence also \( \text{tp}^u((\bar{a}')^\frown (\bar{b}^\prime)/B) = \text{tp}^u((\bar{a}')^\frown \bar{b}/B) \).

Let \( (\bar{c}_i)_{i<\omega} \) be strongly \( A \)-indiscernible such that \( \text{tp}^g(\bar{c}_0/A \cup \{\bar{a}', \bar{b}'\}) \neq \text{tp}^g(\bar{c}_1/A \cup \{\bar{a}', \bar{b}'\}) \) and \( \bar{c}_0, \bar{c}_1 \in D \). By strong indiscernibility, this sequence extends to strongly \( A \)-indiscernible \( (\bar{c}_i)_{i<\omega_1} \). We may assume that \( (\bar{c}_i)_{i<\omega_1} \) is in \( D \) by the previous assumption. Since there are either \( \omega_1 \)-many \( \bar{c}_i \) not realizing \( \text{tp}^g(\bar{c}_0/A \cup \{\bar{a}', \bar{b}'\}) \) or \( \omega_1 \) many \( \bar{c}_i \) not realizing \( \text{tp}^g(\bar{c}_1/A \cup \{\bar{a}', \bar{b}'\}) \), we may assume that
\[
\text{tp}^g(\bar{c}_0/A \cup \{\bar{a}', \bar{b}'\}) \neq \text{tp}^g(\bar{c}_1/A \cup \{\bar{a}', \bar{b}'\})
\]
for each \( i < \omega_1 \).

We claim that \( (\bar{c}_i)_{i<\omega_1} \) has the property that for any \( i_0 < i_1 < \omega_1 \),
\[
\text{tp}^u(\bar{c}_{i_0}, \bar{c}_{i_1}/A \cup \{\bar{a}'\}) = \text{tp}^u(\bar{c}_0, \bar{c}_1/A \cup \{\bar{a}'\}).
\]
Assume, for a contradiction, that there are \( i_0 < i_1 \) such that the above does not hold. We check the following three possibilities:

1. \( 1 < i_0 \)
2. \( i_0 = 0 \) or
3. \( i_0 = 1 \).

Assume that (1) holds. We may skip less than \( \omega_1 \) many tuples if necessary and assume that \( i_0 = 2 \) and \( i_1 = 3 \). The sequence \( (\bar{d}_i)_{i<\omega_1} \), where \( \bar{d}_i = (\bar{c}_0, \bar{c}_1, \alpha, n + \bar{a}') \) for \( i = \alpha + n < \omega_1 \), \( \alpha \) limit and \( n < \omega \), is strongly \( A \)-indiscernible and \( \text{tp}^g(\bar{d}_0/A \cup \{\bar{a}'\}) \neq \text{tp}^g(\bar{d}_1/A \cup \{\bar{a}'\}) \). Then we have that \( \text{tp}^u(\bar{a}'/D) \) Lascar-splits over \( A \), a contradiction. If we have (2), then the sequence \( (\bar{c}_0, \bar{c}_1, i)_{i<\omega_1} \) is strongly \( A \)-indiscernible with \( \text{tp}^u(\bar{c}_0, \bar{c}_1/A \cup \{\bar{a}'\}) \neq \text{tp}^u(\bar{c}_0, \bar{c}_1/A \cup \{\bar{a}'\}) \). We get again that \( \text{tp}^u(\bar{a}'/D) \) Lascar-splits over \( A \), a contradiction. Assume that (1) or (2) does not hold, and thus for all counter-examples \( i_0 < i_1 \) for the claim, \( i_0 = 1 \). We can study the sequence \( (\bar{c}_i)_{i<\omega_1, i \neq 1} \), since \( \text{tp}^g(\bar{c}_0/A \cup \{\bar{a}', \bar{b}'\}) \neq \text{tp}^g(\bar{c}_2/A \cup \{\bar{a}', \bar{b}'\}) \). The claim holds for this sequence.

Now by Corollary 5.9, some \( (\bar{c}_{i_0}, \bar{c}_{i_1}) \) for \( i_0 < i_1 < \omega_1 \) are the beginning of a strongly \( (A \cup \{\bar{a}'\}) \)-indiscernible sequence. By the previous claim we have \( f \in \text{Aut}(\mathfrak{M}'/A \cup \bar{a}') \) mapping \( \bar{c}_{i_0}, \bar{c}_{i_1} \) to \( \bar{c}_0, \bar{c}_1 \) and thus may assume that \( i_0 = 0 \) and
Remark 6.2. We define the property of simplicity, which is not implied by the local character for arbitrary sets. If we add the following definition, we see that under simplicity the bounded closure has a finite character.

We see that under simplicity the bounded closure has a finite character.
Proposition 6.3. Assume that \( (\mathbb{K}, \preceq_K) \) is simple.

1. If \( \text{tp}^w(\bar{a}/B) \) is bounded, then there is finite \( E \subset B \) such that \( \text{tp}^w(\bar{a}/E) \) is bounded.
2. If \( \bar{a} \in \text{bcl}(B) \), then there is finite \( E \subset B \) such that \( \bar{a} \in \text{bcl}(E) \).

Proof. By simplicity, we can choose \( E \subset B \) finite such that \( \bar{a} \downarrow^E B \). Then if \( \text{tp}^w(\bar{a}/B) \) is bounded, also \( \text{tp}^w(\bar{a}/E) \) is bounded by Lemma 5.38(3). If \( \bar{a} \in \text{bcl}(B) \), we have that \( \text{tp}^w(\bar{a}/B) \) is bounded, and thus \( \text{tp}^w(\bar{a}/E) \) is bounded. By Lemma 5.21, \( \bar{a} \in \text{bcl}(E) \).

As a corollary we get that under simplicity bounded closure is a closure operator.

Proposition 6.4. Assume that \( (\mathbb{K}, \preceq_K) \) is simple and \( C \) is a set. Then,

1. \( \text{tp}^w(\bar{a}/C) \) is bounded if and only if \( \bar{a} \in \text{bcl}(C) \) and
2. \( \text{bcl}(C) = \text{bcl}(\text{bcl}(C)) \).

Proof. If \( \text{tp}^w(\bar{a}/C) \) is bounded, we get from the previous Proposition a finite set \( E \subset C \) such that \( \text{tp}^w(\bar{a}/E) \) is bounded. By Lemma 5.21, \( \bar{a} \in \text{bcl}(E) \subset \text{bcl}(C) \). The other direction is clear.

For (2), by Lemma 5.21, it is enough to show that \( \text{bcl}(\text{bcl}(C)) \subset \text{bcl}(C) \). Assume that \( a \in \text{bcl}(\text{bcl}(C)) \). By the previous Proposition, \( a \in \text{bcl}(E') \) for some finite \( E' \subset \text{bcl}(C) \), and furthermore \( E' \subset \text{bcl}(E) \) for some finite \( E \subset C \). Now \( a \in \text{bcl}(E') \subset \text{bcl}(\text{bcl}(E)) = \text{bcl}(E) \), and thus \( a \in \text{bcl}(C) \).

Under simplicity we get a well-behaved independence notion. The proof for the following theorem is identical to the one in [7], using the preliminary results we proved in section 5.1.

Theorem 6.5. Assume that \( (\mathbb{K}, \preceq_K) \) is finitary, simple, stable in \( \aleph_0 \) and has extension property. Then, \( \downarrow \) satisfies the following properties:

1. Invariance: If \( A \downarrow_C B \), then \( f(A) \downarrow_{f(C)} f(B) \) for any \( f \in \text{Aut}(\mathfrak{M}) \).
2. Finite character: \( A \downarrow_C B \) if and only if \( \bar{a} \downarrow_C \bar{b} \) for every finite \( \bar{a} \in A \) and \( \bar{b} \in B \).
3. Monotonicity: If \( A \downarrow_C B \) and \( C \subset D \subset C \cup B \) then \( A \downarrow_C D \) and \( A \downarrow_D B \).
4. Local character: For any finite \( \bar{a} \) and any \( B \) there exists a finite \( E \subset B \) such that \( \bar{a} \downarrow_E B \).
5. Extension: For any \( \bar{a} \), \( C \) and \( B \) containing \( C \) there is \( \bar{b} \) such that \( \text{tp}^w(\bar{b}/C) = \text{tp}^w(\bar{a}/C) \) and \( \bar{b} \downarrow_C B \).
6. For any finite \( C \), \( \bar{a} \) and any \( B \) containing \( C \), there is \( \bar{b} \) such that \( \text{Lstp}(\bar{b}/C) = \text{Lstp}(\bar{a}/C) \) and \( \bar{b} \downarrow_C B \).
7. Symmetry: \( A \downarrow_C B \) if and only if \( B \downarrow_C A \).
(8) **Transitivity:** Let $B \subset C \subset D$. If $A \downarrow B C$ and $A \downarrow C D$, then $A \downarrow B D$.

(9) **Stationarity over $\aleph_0$-saturated models:** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. If $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow \mathcal{A} B$ and $\bar{b} \downarrow \mathcal{A} B$, then $\text{tp}^w(\bar{a}/\mathcal{B} \cup \mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{B} \cup \mathcal{A})$.

(10) **Stationarity of Lascar strong types:** If $\text{Lstp}(\bar{a}/C) = \text{Lstp}(\bar{b}/C)$, $\bar{a} \downarrow C B$ and $\bar{b} \downarrow C B$, then $\text{tp}^w(\bar{a}/\mathcal{B} \cup C) = \text{tp}^w(\bar{a}/\mathcal{B} \cup C)$.

Also the following result follows from these properties and Finite Pairs Lemma 5.40.

**Proposition 6.6** (Pairs Lemma). Assume that $(\mathbb{K}, \preceq_\mathbb{K})$ is simple. Let $A \subset B$, $\bar{a} \downarrow_A B$ and $\bar{b} \downarrow_{(A \cup \{\bar{a}\})} B \cup \{\bar{a}\}$. Then $\bar{a} \preceq \bar{b} \downarrow_A B$.

### 6.1. Simplicity and $U$-rank.

We define $U$-rank in several steps, first over a countable $\aleph_0$-saturated model.

**Definition 6.7.** Let $\mathcal{A}$ be countable and $\aleph_0$-saturated model. Define $U$-rank of $\bar{a}$ over $\mathcal{A}$, $U(\bar{a}/\mathcal{A})$, by induction:

1. **Always** $U(\bar{a}/\mathcal{A}) \geq 0$.
2. $U(\bar{a}/\mathcal{A}) \geq \beta + 1$ if there is countable $\aleph_0$-saturated model $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$, $U(\bar{a}/\mathcal{B}) \geq \beta$ and $\bar{a} \not\equiv^\mathcal{B} \bar{b}$.

For a countable $\aleph_0$-saturated model $\mathcal{A}$, define

$$U(\bar{a}/\mathcal{A}) = \min\{\alpha : U(\bar{a}/\mathcal{A}) \not\geq \alpha + 1\}$$

if such an ordinal exists. Then define $U$-rank for arbitrary $\aleph_0$-saturated model $\mathcal{A}$ as

$$U(\bar{a}/\mathcal{A}) = \min\{U(\bar{a}/\mathcal{A}'): \mathcal{A}' \subset \mathcal{A} \text{ countable $\aleph_0$-saturated model}\}$$

The $U$-rank of $\bar{a}$ over a countable $\aleph_0$-saturated $\mathcal{A}$ is always defined, since by Theorem 3.16 there cannot be an infinite chain of models $\mathcal{A}_i$ such that $\bar{a} \downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$ for each $i < \omega$. Also for countable $\aleph_0$-saturated $\mathcal{A}$, $U(\bar{a}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}')$ for all $\aleph_0$-saturated $\mathcal{A}' \subset \mathcal{A}$ by (2) of the following remark. These two remarks follow easily from the definition.

**Remark 6.8.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\aleph_0$-saturated models

1. If $U(\bar{a}/\mathcal{A}) = \alpha$ and $g$ is an automorphism of $\mathcal{M}$ then $U(g(\bar{a})/g(\mathcal{A})) = \alpha$.
2. If $\mathcal{A} \subset \mathcal{B}$, then $U(\bar{a}/\mathcal{B}) \leq U(\bar{a}/\mathcal{A})$.

**Definition 6.9.** We say that $\bar{a}$ and a set $A$ are finitely equivalent to $\bar{a}'$ and $A'$, write

$$(\bar{a}, A) \equiv_0 (\bar{a}', A')$$

if there is a bijective mapping $f : \bar{a} \cup A \rightarrow \bar{a}' \cup A'$ such that $f(\bar{a}) = \bar{a}'$ and for each $\bar{b} \in A$ $\text{tp}^g(\bar{a} \cup \bar{b}/\emptyset) = \text{tp}^g(\bar{a}' \cup f(\bar{b})/\emptyset)$. 
We see that if \( \text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{a}'/A) \), then \((\bar{a}, A) \equiv_0 (\bar{a}', A)\).

**Remark 6.10.** If \( \mathcal{A} \) and \( \mathcal{A}' \) are \( \aleph_0 \)-saturated models such that \((\bar{a}, \mathcal{A}) \equiv_0 (\bar{a}', \mathcal{A}')\), then \( U(\bar{a}/\mathcal{A}) = U(\bar{a}'/\mathcal{A}') \).

**Proof.** By the definition of \( U \)-rank, it is enough to prove the claim for all countable \( \mathcal{A} \) and \( \mathcal{A}' \). Hence we assume that \( \mathcal{A} \) and \( \mathcal{A}' \) are countable.

Let \( f : \bar{a} \cup \mathcal{A} \rightarrow \bar{a}' \cup \mathcal{A}' \) be the mapping from the definition 6.9. Now \( f \upharpoonright \mathcal{A} : \mathcal{A} \rightarrow \mathcal{A}' \) extends to an automorphism \( g \).

When \( \bar{c} \in \mathcal{A}' \) finite, we have that \( g^{-1}(\bar{c}) = f^{-1}(\bar{c}) \in \mathcal{A} \) and \( \text{tp}^g(\bar{a}\\bar{c}/\emptyset) = \text{tp}^g(\bar{a}\\f^{-1}(\bar{c})/\emptyset) = \text{tp}^g(\bar{a}'\\bar{c}/\emptyset) \). Thus \( \text{tp}^w(g(\bar{a})/\mathcal{A}') = \text{tp}^w(\bar{a}'/\mathcal{A}') \) and we get from Theorem 3.12 an automorphism \( h \) such that \( h(\bar{a}) = \bar{a}' \) and \( h \upharpoonright \mathcal{A} = \text{id}_{\mathcal{A}'} \).

Now \( h \circ g \) is an automorphism, \( h \circ g(\bar{a}) = \bar{a}' \) and \( h \circ g(\mathcal{A}) = \mathcal{A}' \). The claim follows from Remark 6.8(1).

We prove that when \( \mathcal{A} \subset \mathcal{B} \) are countable \( \aleph_0 \)-saturated models, \( U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B}) \) if and only if \( \bar{a} \downarrow_{\mathcal{A}} \mathcal{B} \).

**Lemma 6.11.** Assume that \( \bar{a} \downarrow_{\mathcal{A}} \mathcal{B}, \mathcal{A} \subset \mathcal{B} \) and \( \mathcal{A}, \mathcal{B} \) are countable, \( \aleph_0 \)-saturated models. Then if \( U(\bar{a}/\mathcal{A}) \geq \alpha \), also \( U(\bar{a}/\mathcal{B}) \geq \alpha \).

**Proof.** The proof is by induction on \( \alpha \), and we prove the implication for all \( \mathcal{A}, \mathcal{B} \) and \( \bar{a} \) simultaneously. If \( \alpha \) is 0 or a limit ordinal, the induction step is clear. Assume that \( \alpha = \beta + 1 \) and that \( \mathcal{C} \) is an \( \aleph_0 \)-saturated countable model such that \( \mathcal{A} \subset \mathcal{C}, \bar{a} \downarrow_{\mathcal{A}} \mathcal{C}, \) and \( U(\bar{a}/\mathcal{C}) \geq \beta \).

We use Lemma 3.18 to get a tuple \( \bar{a}' \) and countable set \( \mathcal{C}' \) such that \( \text{tp}^w(\bar{a}'/\mathcal{A}') = \text{tp}^w(\bar{a}'\mathcal{C}'/\mathcal{A}) \), \( \bar{a}' \mathcal{C}' \downarrow_{\mathcal{A}'} \mathcal{B} \). Then also \( (\bar{a}', \mathcal{C}') \equiv_0 (\bar{a}', \mathcal{C}) \). Since we may write an automorphism mapping \( \mathcal{C} \) to \( \mathcal{C}' \), we see that also \( \mathcal{C}' \) is an \( \aleph_0 \)-saturated model. Then from Remark 6.10 we get that \( U(\bar{a}'/\mathcal{C}') \geq \beta \). Also \( \mathcal{A} \subset \mathcal{C}', \text{tp}^w(\bar{a}'/\mathcal{A}) = \text{tp}^w(\bar{a}'/\mathcal{A}') \) and we can also easily see that \( \bar{a}' \downarrow_{\mathcal{A}'} \mathcal{C}' \).

Let \( \mathcal{D} \) be a countable \( \aleph_0 \)-saturated model such that \( \mathcal{C}' \cup \mathcal{B} \subset \mathcal{D} \). From countable extension we get \( \bar{a}^* \) such that \( \text{tp}^w(\bar{a}^*/\mathcal{C}') = \text{tp}^w(\bar{a}'/\mathcal{C}') \) and \( \bar{a}^* \downarrow_{\mathcal{D}} \mathcal{B} \). Then also \( \mathcal{C}' \subset \mathcal{D} \) and \( U(\bar{a}^*/\mathcal{C}') = U(\bar{a}'/\mathcal{C}') \geq \beta \), and by induction hypothesis,

\[
U(\bar{a}^*/\mathcal{D}) \geq \beta.
\]

Next we would like to show that \( \text{tp}^w(\bar{a}^*/\mathcal{B}) = \text{tp}^w(\bar{a}/\mathcal{B}) \). In order to do that, we take arbitrary finite \( \bar{b} \in \mathcal{B} \) and claim that

\[
\bar{a}^* \downarrow_{\mathcal{B}} \bar{b}.
\]

Let \( \bar{b}' \) be a free extension such that \( \text{tp}^w(\bar{b}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A}) \) and \( \bar{b}' \downarrow_{\mathcal{B}} \mathcal{C}' \). Let \( \bar{c} \in \mathcal{C}' \) finite. Since \( \mathcal{C}' \downarrow_{\mathcal{B}} \), we get from symmetry that \( \bar{b} \downarrow_{\mathcal{B}} \bar{c} \). By monotonicity
$\bar{b} \downarrow^s_{cf} \bar{c}$ and we get from stationarity that $\text{tp}^w(\bar{b}/\mathcal{A} \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/\mathcal{A} \cup \{\bar{c}\})$. Since this holds for each finite $\bar{c} \in \mathcal{C}'$, we get that $\text{tp}^w(\bar{b}/\mathcal{C}') = \text{tp}^w(\bar{b}/\mathcal{C}')$. Then also $\bar{b} \downarrow^s_{cf} \mathcal{C}'$.

Since $\bar{a}^* \downarrow^s_{cf} \mathcal{D}$, we get that $\bar{a}^* \downarrow^s_{cf} \bar{b}$ and again from symmetry that $\bar{b} \downarrow^s_{cf} \bar{a}^*$. Now we have that $\mathcal{A} \subset \mathcal{C}' \subset \mathcal{C}' \cup \{\bar{a}^*\}$, $\mathcal{C}' \aleph_0$-saturated, $\bar{b} \downarrow^s_{cf} \mathcal{C}' \cup \{\bar{a}^*\}$ and $\bar{b} \downarrow^s_{cf} \mathcal{C}'$. We may use transitivity to get $\bar{b} \downarrow^s_{cf} \mathcal{C}' \cup \{\bar{a}^*\}$. Claim (6) follows from symmetry.

Now we take a free extension $\bar{d}$ such that $\text{tp}^w(\bar{d}/\mathcal{A}) = \text{tp}^w(\bar{a}^*/\mathcal{A})$ and $\bar{d} \downarrow^s_{cf} \mathcal{B}$. Then from (6) we get that for each finite $\bar{b} \in \mathcal{B}$ both $\bar{d} \downarrow^s_{cf} \bar{b}$ and $\bar{a}^* \downarrow^s_{cf} \bar{b}$. Again we get from stationarity that $\text{tp}^w(\bar{a}^*/\mathcal{A} \cup \bar{b}) = \text{tp}^w(\bar{d}/\mathcal{A} \cup \bar{b})$ for each finite $\bar{b} \in \mathcal{B}$, and thus $\text{tp}^w(\bar{a}^*/\mathcal{B}) = \text{tp}^w(\bar{d}/\mathcal{B})$. Hence also $\bar{a}^* \downarrow^s_{cf} \mathcal{B}$.

Then since $\bar{a}^* \downarrow^s_{cf} \mathcal{B}$, $\bar{a}^* \downarrow^s_{cf} \mathcal{B}$ and $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{a}^*/\mathcal{A})$, we again get from stationarity that

$$\text{tp}^w(\bar{a}/\mathcal{B}) = \text{tp}^w(\bar{a}^*/\mathcal{B}).$$

Since we have that $\mathcal{B} \subset \mathcal{D}$, $\mathcal{D} \aleph_0$-saturated and we have shown (5), we would like to show that also

$$\bar{a}^* \not\downarrow^s_{cf} \mathcal{D}.\tag{8}$$

Assume the contrary, that $\bar{a}^* \downarrow^s_{cf} \mathcal{D}$. Then we get from (7) and $\bar{a} \downarrow^s_{cf} \mathcal{B}$ that $\bar{a}^* \downarrow^s_{cf} \mathcal{B}$ and furthermore from transitivity that $\bar{a}^* \downarrow^s_{cf} \mathcal{D}$. But then since $\mathcal{C}' \subset \mathcal{D}$, also $\bar{a}^* \downarrow^s_{cf} \mathcal{C}'$. This is a contradiction, since we chose $\bar{a}^*$ so that $\text{tp}^w(\bar{a}^*/\mathcal{C}') = \text{tp}^w(\bar{a}/\mathcal{C}')$ and we know that $\bar{a}^* \not\downarrow^s_{cf} \mathcal{C}'$.

We have now that

$$U(\bar{a}^*/\mathcal{B}) \geq \alpha.\tag{9}$$

Then finally from (9), (7) and Remark 6.10 we get that $U(\bar{a}/\mathcal{B}) \geq \alpha$. \hfill \Box

**Theorem 6.12.** For $\aleph_0$-saturated models $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$, $\bar{a} \downarrow^s_{cf} \mathcal{B}$ if and only if $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$.

**Proof.** We prove the claim first for countable $\mathcal{A}$ and $\mathcal{B}$. If $\bar{a} \not\downarrow^s_{cf} \mathcal{B}$, we can take $\mathcal{B}$ in Definition 6.7 to show that $U(\bar{a}/\mathcal{A}) \geq U(\bar{a}/\mathcal{B}) + 1$. Thus from $U(\bar{a}/\mathcal{A}) = U(\bar{b}/\mathcal{B})$ it follows that $\bar{a} \downarrow^s_{cf} \mathcal{B}$. Also if we have that $\bar{a} \downarrow^s_{cf} \mathcal{B}$, we get from Lemma 6.11 that $U(\bar{a}/\mathcal{A}) \leq U(\bar{a}/\mathcal{B})$, and then by 6.8(2) $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$.

Then let $\mathcal{A}$ and $\mathcal{B}$ be of arbitrary size. Assume that $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$. Let $\mathcal{B}' \subset \mathcal{B}$ be a countable $\aleph_0$-saturated model such that $U(\bar{a}/\mathcal{B}')$ is minimal. Then there must be some countable $\aleph_0$-saturated $\mathcal{A}' \subset \mathcal{A}$ such that $U(\bar{a}/\mathcal{A}') = U(\bar{a}/\mathcal{B}')$. Now if $\bar{a} \not\downarrow^s_{cf} \mathcal{B}'$, also $\bar{a} \not\downarrow^s_{cf} \mathcal{B}$ and we can find a countable $\aleph_0$-saturated model $\mathcal{B}'' \subset \mathcal{B}$ such that $\mathcal{A}' \cup \mathcal{B} \subset \mathcal{B}''$ and $\bar{a} \not\downarrow^s_{cf} \mathcal{B}''$. Now $U(\bar{a}/\mathcal{B}'') \neq U(\bar{a}/\mathcal{B})$.
$U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$ and since $\mathcal{B} \subset \mathcal{B}'$, $U(\bar{a}/\mathcal{B}') \leq U(\bar{a}/\mathcal{B})$. This contradicts the minimality of $U(\bar{a}/\mathcal{B}')$. Thus from $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$ we get that $\bar{a} \downarrow^*_{\mathcal{A}} \mathcal{B}$.

Then assume that $\bar{a} \downarrow^*_{\mathcal{A}} \mathcal{B}$. Let $\mathcal{A}'$ be a countable $\omega$-saturated model such that $\bar{a} \downarrow^*_{\mathcal{A}'} \mathcal{B}$ and furthermore $\mathcal{A}' \models \text{Lemma 5.11}$. Then let $\mathcal{B}'$ be a countable $\aleph_0$-saturated model such that $\mathcal{A}' \cup \mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$. Now since $\bar{a} \downarrow^*_{\mathcal{A}'} \mathcal{B}'$, we have that $U(\bar{a}/\mathcal{A}') = U(\bar{a}/\mathcal{B}')$. Then since $\mathcal{B}' \subset \mathcal{B}''$, we have that $U(\bar{a}/\mathcal{B}'') \leq U(\bar{a}/\mathcal{B}')$, and thus $U(\bar{a}/\mathcal{B}'') = U(\bar{a}/\mathcal{B})$. We get that $U(\bar{a}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}') = U(\bar{a}/\mathcal{B})$, and since $\mathcal{A} \subset \mathcal{B}$, $U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})$. \hfill \qed

We now define $U$-rank of $\text{tp}^w(\bar{a}/A)$, where $A$ is an ordinary set.

\textbf{Definition 6.13.} For finite $\bar{a}$ and a set $A$, define

$$U(\bar{a}/A) = \sup\{U(\bar{b}/\mathcal{A}) : \text{tp}^w(\bar{b}/A) = \text{tp}^w(\bar{a}/A), \text{ } A \subset \mathcal{A} \text{ and } \mathcal{A} \aleph_0 \text{-saturated}\}.$$  

We can prove the following using the properties of $U$-rank over $\aleph_0$-saturated models.

\textbf{Lemma 6.14.} Let $E$ be finite.

\begin{enumerate}
  
  \item[(1)] Let $\mathcal{A}$ be an $\aleph_0$-saturated model containing $E$. Then
  $$U(\bar{a}/E) = \sup\{U(\bar{b}/\mathcal{A}) : \text{tp}^w(\bar{b}/E) = \text{tp}^w(\bar{a}/E)\}.$$ 

  \item[(2)] $U(\bar{a}/E) = \sup\{\text{tp}^w(\bar{a}/\mathcal{A}) : E \subset \mathcal{A}, \text{ } \mathcal{A} \text{ } \omega \text{-saturated}\}$.

\end{enumerate}

Finally we define extensible $U$-rank.

\textbf{Definition 6.15 (Extensible U-rank).} We say that $(\mathbb{K}, \prec_{\mathbb{K}})$ has extensible $U$-rank if for each finite $E$, each $\bar{a}$ and each $\aleph_0$-saturated model $\mathcal{A}$ containing $E$ there is $\bar{b}$ such that $\text{tp}^w(\bar{a}/E) = \text{tp}^w(\bar{b}/E)$ and $U(\bar{b}/\mathcal{A}) = U(\bar{a}/E)$.

\textbf{Proposition 6.16.} Assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ has extensible $U$-rank. Then, for each finite $E$ and $\bar{a}$, we have $\bar{a} \downarrow_E E$.

\textbf{Proof.} Let $\mathcal{A}$ be an $\aleph_0$-saturated model containing $E$ and $\mathcal{A}$ a set of strongly $E$-indiscernible sequences satisfying the properties (1), (2) and (3) of Lemma 5.11. Since $(\mathbb{K}, \prec_{\mathbb{K}})$ has extensible $U$-rank, we can choose $\bar{b}$ such that $\text{tp}^w(\bar{b}/E) = \text{tp}^w(\bar{a}/E)$ and $U(\bar{b}/\mathcal{A}) = U(\bar{a}/E)$. It is enough to show that $\text{tp}^w(\bar{b}/\mathcal{A})$ does not Lascar-split over $E$. If we show that, from Proposition 5.34 it follows that $\bar{b} \downarrow_E \mathcal{A}$, and furthermore $\bar{b} \downarrow_E E$ by monoticity and $\bar{a} \downarrow_E E$, since $\text{tp}^w(\bar{b}/E) = \text{tp}^w(\bar{a}/E)$.

Assume, for a contradiction, that $(\bar{a}_i)_{i<\omega}$ is strongly $E$-indiscernible with $\bar{a}_0, \bar{a}_i \in \mathcal{A}$ and $\text{tp}^w(\bar{a}_0/E \cup \{\bar{b}\}) \neq \text{tp}^w(\bar{a}_i/E \cup \{\bar{b}\})$. Now by property (2) of Lemma 5.11 we may assume that $(\bar{a}_i)_{i<\omega} \subset \mathcal{A}$ and $(\bar{a}_i)_{i<\omega} \in \mathcal{A}$. Furthermore by (1) of Lemma 5.11 we may assume that

$$\text{tp}^0(\bar{a}_i/E \cup \{\bar{b}\}) \neq \text{tp}^0(\bar{a}_0/E \cup \{\bar{b}\}) \text{ for all } 0 < i < \omega_1.$$
By assumption (3) of Lemma 5.11 on $\mathcal{A}$, for each $i < \omega_1$, there is $g_i \in \text{Aut}(\mathcal{A}/E)$ such that $g_i(\bar{a}_k) = \bar{a}_{i+k}$ for each $k < \omega_1$. Then since $\mathcal{A}$ is a model, for each $i < \omega_1$ there is $f_i \in \text{Aut}(\mathcal{M}/E)$ extending $g_i$. Let $\bar{b}_i = f_i(\bar{b})$ for each $i < \omega_1$. We get that\
tp^w(\bar{b}_i/\mathcal{A}) \neq \tp^w(\bar{b}_j/\mathcal{A})$, when $i < j < \omega_1$.

Since if $i < j$ and there would be $h \in \text{Aut}(\mathcal{M}/E \cup \{\bar{a}_j\})$ such that $h(\bar{b}_j) = \bar{b}_i$, we would have that $(f_i^{-1} \circ h \circ f_j)(\bar{b}) = \bar{b}$ and $(f_i^{-1} \circ h \circ f_j)(\bar{a}_0) = \bar{a}_k$, $k > 0$, a contradiction.

Let $\mathcal{A}_0 \preceq_{\mathcal{K}} \mathcal{A}$ be $\aleph_0$-saturated and countable such that $E \subset \mathcal{A}_0$. An automorphism preserves $U$-rank, and thus $U(\bar{b}_i/\mathcal{A}) = U(\bar{b}_i/E)$ for each $i < \omega_1$. Also since $\mathcal{A}_0 \subset \mathcal{A}$, we have that $U(\bar{b}_i/\mathcal{A}) \leq U(\bar{b}_i/\mathcal{A}_0)$, and since $\mathcal{A}_0$ is an $\aleph_0$-saturated model containing $E$, we get from the definition of $U(\bar{b}_i/E)$ that $U(\bar{b}_i/\mathcal{A}_0) \leq U(\bar{b}_i/E)$. Thus $U(\bar{b}_i/\mathcal{A}) = U(\bar{b}_i/\mathcal{A}_0)$ and hence by Theorem 6.12

$\bar{b}_i \downarrow A_0$ for all $i < \omega_1$.

By stationarity of types over $\aleph_0$-saturated models (Theorem 6.5(8)) we must have that

$\tp^w(\bar{b}_i/\mathcal{A}_0) \neq \tp^w(\bar{b}_j/\mathcal{A}_0)$ for all $i < j < \omega_1$.

Since $\mathcal{A}_0$ is countable, this contradicts $\aleph_0$-stability. \hfill $\square$

Finally, as in [7], from Proposition 6.16 we get the following result.

**Theorem 6.17.** Assume that $(\mathcal{K}, \preceq_{\mathcal{K}})$ is a finitary AEC, stable in $\aleph_0$, with extension property and extensible $U$-rank. Then $(\mathcal{K}, \preceq_{\mathcal{K}})$ is simple.

We say that $(\mathcal{K}, \preceq_{\mathcal{K}})$ has finite $U$-rank, if for each finite sequence $\bar{a}$,

$\sup\{U(\bar{a}/\mathcal{A}) : \mathcal{A} \in \mathcal{K} \text{ countable and } \aleph_0\text{-saturated} \} < \aleph_0$.

We can also establish the usual addition properties for $U$-rank, and so it is enough to check the above condition for each singleton $a$.

As a corollary of Theorem 6.17 we get the following. Here tameness could be replaced with categoricity above the Hanf number.

**Corollary 6.18.** Assume that $(\mathcal{K}, \preceq_{\mathcal{K}})$ is a tame finitary AEC, stable in $\aleph_0$, with finite $U$-rank. Then $(\mathcal{K}, \preceq_{\mathcal{K}})$ is simple.

**References**


PAPER II

CATEGORICITY TRANSFER IN SIMPLE FINITARY ABSTRACT ELEMENTARY CLASSES

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Submitted.
CATEGORICITY TRANSFER IN SIMPLE FINITARY ABSTRACT ELEMENTARY CLASSES

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Abstract. We continue to study finitary abstract elementary classes, defined in [6]. We remark that we are able to obtain the results of [6] in a more general setting, replacing the assumptions of disjoint amalgamation and a prime model with ordinary amalgamation and joint embedding. We introduce a concept of weak $\kappa$-categoricity and an $\mathfrak{f}$-primary model in an $\aleph_0$-stable simple finitary AEC with the extension property, and gain the following theorem: Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be a simple finitary AEC, weakly categorical in some uncountable $\kappa$. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is weakly categorical in each $\lambda \geq \min\{\kappa, \beth_{\mathfrak{f}}(\aleph_0)^+\}$. We show that if the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is also LS$(\mathbb{K})$-tame, weak $\kappa$-categoricity is equivalent with $\kappa$-categoricity in the usual sense.

1. Introduction

This paper continues the study of finitary abstract elementary classes from [6], where our aim was to formulate a general context for studying geometric stability theory in nonelementary classes. In [6], we studied an abstract elementary class $(\mathbb{K}, \preceq_{\mathbb{K}})$ with LS$(\mathbb{K}) = \aleph_0$, arbitrariry large models, disjoint amalgamation, a prime model and a property we called finite character. Finite character generalizes the following property of first order logic: for two models $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$ if and only if $\text{tp}(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}(\bar{b}/\emptyset, \mathcal{B})$ for all finite tuples $\bar{a} \in \mathcal{A}$. We introduced weak types and studied them instead of the usual notion of Galois types in abstract elementary classes. We also showed that under tameness and $\aleph_0$-stability the two notions coincide. In [6], we also defined a notion of independence for an $\aleph_0$-stable finitary abstract elementary class and showed that under simplicity and the extension property it has all the usual properties of first order non-forking of complete types. Simplicity is needed to gain the properties over sets, not only over $\aleph_0$-saturated models. We also proved that the extension property is implied

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either by tameness or categoricity above the Hanf number, and that simplicity in this case is implied by finite $U$-rank.

In this paper the concept of a finitary class is slightly more general than the one we study in [6]. Instead of disjoint amalgamation and a prime model we assume amalgamation and joint embedding. We notice that we gain most of the results in [6] with these slightly weaker assumptions. The main difference is the construction of the monster model. In [6], the monster model was a reduct of a homogeneous model $\mathfrak{M}^*$ in the extended language $\tau^*$. In this paper the monster model is again a reduct of a model in the language $\tau^*$, but it does not need to be homogeneous. This approach is similar to the original Shelah’s Presentation Theorem, see [14]. Except for the existence of such homogeneous model, all the main results of [6] can be proved and for example strongly indiscernible sequences can be found. In section 2 we reprove results that in [6] used the assumptions of disjoint amalgamation and a prime model. Proposition 2.13 collects the information needed to replace the use of the homogeneous model $\mathfrak{M}^*$ in [6]. Theorem 4.3 of [6] about symmetry is here Theorem 2.33, Proposition 4.4 and Proposition 4.19 about the extension property are here Proposition 2.34 and Theorem 2.31. Corollary 5.9 of [6] about finding strongly indiscernible sequences is here stated as Corollary 2.50. We also prove a (slightly weaker) version of Lemma 5.20 as Lemma 2.51, which shows that the bounded closure of an $\aleph_0$-saturated model is the model itself.

We introduce two notions of a primary model in $\aleph_0$-stable finitary classes with the extension property. A primary model $\mathscr{A}[\bar{a}]$ is a constructible model over a tuple $\bar{a}$ and a countable $\aleph_0$-saturated model $\mathscr{A}$. An $f$-primary model is constructible over $\mathscr{A} \cup B$, where $\mathscr{A}$ is an $\aleph_0$-saturated model and $B$ is an arbitrary set. The existence of $f$-primary models over $\mathscr{A} \cup B$, where $B$ is a set, is implied by simplicity. We introduce also a concept of weak $\kappa$-categoricity, which means that each model of size $\kappa$ is weakly saturated, and use $f$-primary models to show the following theorem (Theorem 4.10):

**Theorem 1.1.** Assume that $(\mathbb{K}, \preceq_\mathbb{K})$ is a simple finitary AEC which is weakly categorical in some uncountable cardinal $\kappa$. Then $((\mathbb{K})^\omega, \preceq_\mathbb{K})$ is weakly categorical in each uncountable $\kappa$ and $(\mathbb{K}, \preceq_\mathbb{K})$ is weakly categorical in each $\lambda$ such that $\lambda \geq \min\{\kappa, \beth_{2^{\aleph_0}+}\}$.

Here $(\mathbb{K})^\omega$ is the class of $\aleph_0$-saturated models of $\mathbb{K}$. As a corollary we get the following (Corollary 4.14):

**Corollary 1.2.** Assume that $(\mathbb{K}, \preceq_\mathbb{K})$ is a simple LS($\mathbb{K}$)-tame finitary AEC categorical in some uncountable $\kappa$. Then $((\mathbb{K})^\omega, \preceq_\mathbb{K})$ is categorical in each uncountable $\kappa$ and $(\mathbb{K}, \preceq_\mathbb{K})$ is categorical in each $\lambda$ such that $\lambda \geq \min\{\kappa, \beth_{2^{\aleph_0}+}\}$. 
The last corollary is another partial result towards a categoricity transfer for abstract elementary classes. In [13], Shelah showed that for an excellent class, categoricity in some uncountable \( \kappa \) implies categoricity in each uncountable \( \kappa \). For a study on excellent classes, see for example [9]. Our class generalizes the excellent setting i.e. atomic \( \aleph_0 \)-stable excellent classes are \( \aleph_0 \)-stable tame finitary AEC’s and every model is \( \aleph_0 \)-saturated. Other studies of categoricity in more general settings can be found for example in [15], [16], [5] and [2].

In [15], Shelah shows that for an abstract elementary class with amalgamation and joint embedding, there is a cardinal \( \chi_0 \) called the second Hanf number, such that if \( \chi_0 < \lambda \leq \kappa \) and the class is categorical in the successor cardinal \( \kappa^+ \), then it is categorical in \( \lambda \). In the book [1] Baldwin shows that we can lower the value of \( \chi_0 \) to the the actual Hanf number, which is \( 2^{LS(\kappa)} \). This implies that from any upward categoricity transfer we gain categoricity at least for all \( \kappa \) above the Hanf number. In [4], Grossberg and VanDieren separated the notion of tame classes as a useful and quite general setting to study the upward categoricity transfer. The best result so far would be in [3], where the context is an abstract elementary class with amalgamation, joint embedding, arbitrary large models and tameness in some cardinal \( \chi \). Then categoricity in some successor \( \kappa^+ > \max\{\chi, LS(\kappa)\} \) implies categoricity in all \( \lambda \geq \kappa^+ \). Also the case when \( \kappa^+ = LS(\kappa) \) is studied by Grossberg and VanDieren in [3], but they need to assume categoricity in \( LS(\kappa) \). Lessmann shows in [11] that upward categoricity is gained assuming categoricity in \( R_1 = \chi^+ = LS(\kappa) \). Our result does not obtain such generality, but we are able to give up the assumption that the categoricity cardinal should be a successor. Also our method of simplicity and primary models is different.

2. Finitary abstract elementary classes

Let \( \tau \) be a countable vocabulary. We recall the definitions of an abstract elementary class, amalgamation, joint embedding and finite character.

**Definition 2.1.** A class of \( \tau \)-structures \( (\mathbb{K}, \preceq) \) is an abstract elementary class if

1. Both \( \mathbb{K} \) and the binary relation \( \preceq \) are closed under isomorphism.
2. If \( \mathcal{A} \preceq \mathcal{B} \), then \( \mathcal{A} \) is a substructure of \( \mathcal{B} \).
3. \( \preceq \) is a partial order on \( \mathbb{K} \).
4. If \( \langle \mathcal{A}_i : i < \delta \rangle \) is an \( \preceq \)-increasing chain, then
   a. \( \bigcup_{i < \delta} \mathcal{A}_i \in \mathbb{K} \);
   b. for each \( j < \delta \), \( \mathcal{A}_j \preceq \bigcup_{i < \delta} \mathcal{A}_i \);
   c. if each \( \mathcal{A}_i \preceq \mathcal{M} \in \mathbb{K} \), then \( \bigcup_{i < \delta} \mathcal{A}_i \preceq \mathcal{M} \).
5. If \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K} \), \( \mathcal{A} \preceq \mathcal{C} \), \( \mathcal{B} \preceq \mathcal{C} \) and \( \mathcal{A} \subseteq \mathcal{B} \) then \( \mathcal{A} \preceq \mathcal{B} \).
There is a Löwenheim-Skolem number $\text{LS}(K)$ such that if $\mathcal{A} \in K$ and $B \subset \mathcal{A}$ a subset, there is $\mathcal{A}' \in K$ such that $B \subset \mathcal{A}' \preceq_K \mathcal{A}$ and $|\mathcal{A}'| = |B| + \text{LS}(K)$.

When $\mathcal{A} \preceq_K B$, we say that $B$ is a $K$-extension of $\mathcal{A}$ and $\mathcal{A}$ is a $K$-submodel of $B$. If $\mathcal{A}, B \in K$ and $f : \mathcal{A} \to B$ an embedding such that $f(\mathcal{A}) \preceq_K B$, we say that $f$ is a $K$-embedding.

**Definition 2.2 (Amalgamation).** We say that $(K, \preceq_K)$ has the amalgamation property, if it satisfies the following:

If $\mathcal{A}, B, C \in K$, $\mathcal{A} \preceq_K B$, $\mathcal{A} \preceq_K C$ and $B \cap C = \mathcal{A}$, there is $D \in K$ and a map $f : B \cup C \to D$ such that $f \upharpoonright B$ and $f \upharpoonright C$ are $K$-embeddings.

**Definition 2.3 (Joint embedding).** We say that $(K, \preceq_K)$ has the joint embedding property if for every $\mathcal{A}, B \in K$ there is $C \in K$ and $K$-embeddings $f : \mathcal{A} \to C$ and $g : B \to C$.

To define finite character we use the following concept of $\mathcal{A}$-Galois type.

**Definition 2.4 ($\mathcal{A}$-Galois type).** For $\mathcal{A}, B \in K$ and $\bar{a} \in \mathcal{A}, \bar{b} \in B$ we say

$$\text{tp}^g(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^g(\bar{b}/\emptyset, \mathcal{B})$$

if there is $C \in K$ and $K$-embeddings $f : \mathcal{A} \to C$ and $g : B \to C$ such that $f(\bar{a}) = g(\bar{b})$.

With finite character, we can decide whether a model is a $K$-submodel of another model by only looking at all finite parts of it.

**Definition 2.5 (Finite character).** We say that an AEC $(K, \preceq_K)$ has finite character, if it satisfies the following: If, $\mathcal{A}, B \in K$, $\mathcal{A} \subseteq B$, and for each finite tuple $\bar{a} \in \mathcal{A}$ we have $\text{tp}^g(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^g(\bar{a}/\emptyset, \mathcal{B})$, then $\mathcal{A} \preceq_K B$.

Finite character implies that if $\mathcal{A} \preceq_K B$ and $f : \mathcal{A} \to B$ is a mapping, then $f$ is a $K$-embedding if and only if

$$\text{tp}^g(\bar{a}/\emptyset, \mathcal{B}) = \text{tp}^g(f(\bar{a})/\emptyset, \mathcal{B})$$

for each $\bar{a} \in \mathcal{A}$.

In [6], the authors defined a finitary abstract elementary class to be an abstract elementary class with arbitrary large models, $\text{LS}(K)$ being $\aleph_0$, disjoint amalgamation, a prime model over $\emptyset$ and finite character. In this paper we study a classes with weaker assumptions, but still call them finitary abstract elementary classes, since almost all the main theorems of [6] hold for these classes also. In this section we show how these results can be proved with the weaker assumptions, but skip
those proofs which will be exactly the same. The main difference is that we are not able to construct a monster model with as good homogeneity properties as in [6].

**Definition 2.6** (Finitary abstract elementary class). We say that an abstract elementary class \((\mathcal{K}, \preceq_{\mathcal{K}})\) is finitary, if it satisfies the following:

1. \(\text{LS}(\mathcal{K}) = \aleph_0\).
2. \((\mathcal{K}, \preceq_{\mathcal{K}})\) has arbitrarily large models,
3. \((\mathcal{K}, \preceq_{\mathcal{K}})\) has the amalgamation property,
4. \((\mathcal{K}, \preceq_{\mathcal{K}})\) has the joint embedding property and
5. \((\mathcal{K}, \preceq_{\mathcal{K}})\) has finite character.

From now on we will assume that \((\mathcal{K}, \preceq_{\mathcal{K}})\) is a finitary abstract elementary class. We will also consider \(\aleph_0\)-stability and the extension property. Both of these will be implied by \(\kappa\)-categoricity above the Hanf number \(H\) of abstract elementary classes with \(\text{LS}(\mathcal{K}) = \aleph_0\). Especially, we define that \(H = \beth(2^{\aleph_0})^+\), although this number is only known to be an upper bound for the Hanf number.

In [6], we proved a stronger version of the representation theorem of Shelah’s using disjoint amalgamation and prime model. Here we use a version which is a special case of the original one in [14]. The reader should look for details in [6] or for even more detailed proofs in [18].

**Definition 2.7.** For \(n, k < \omega\), let \(F_k^n\) be a \(k\)-ary function symbol, \(\tau^* = \tau \cup \{F_k^n : n, k < \omega\}\) and \(\mathcal{K}^*\) be the class of all \(\tau^*\)-structures such that for \(A \in \mathcal{K}^*\):

1. \(A \upharpoonright \tau \in \mathcal{K}\),
2. For all \(\bar{a} \in \mathcal{A}\), \(\mathcal{A}_{\bar{a}} = \{(F_n^{d(\bar{a})})_{\mathcal{A}}(\bar{a}) : 0 < n < \omega\}\), is such that
   (a) \(\mathcal{A}_{\bar{a}} \in \mathcal{K}\) and \(\mathcal{A}_{\bar{a}} \preceq_{\mathcal{K}} \mathcal{A} \upharpoonright \tau\),
   (b) if \(\bar{b} \subseteq \bar{a}\) then \(\bar{b} \in \mathcal{A}_{\bar{a}} \subseteq \mathcal{A}_{\bar{a}}^1\).

**Lemma 2.8.** (Shelah) If \(\mathcal{A} \in \mathcal{K}^*\) and \(B \subseteq \mathcal{A}\) a subset such that \(B\) is closed under functions \(F_n^k\), then \(B \upharpoonright \tau \in \mathcal{K}\) and \(B \upharpoonright \tau \preceq_{\mathcal{K}} \mathcal{A} \upharpoonright \tau\).

**Lemma 2.9.** (Shelah) For every \(\mathcal{A} \in \mathcal{K}\) there is \(\mathcal{A}^* \in \mathcal{K}^*\) such that \(\mathcal{A}^* \upharpoonright \tau = \mathcal{A}\). Furthermore, if \(\mathcal{A}_0 \preceq_{\mathcal{K}} \mathcal{A}\) we can choose \(\mathcal{A}^*\) and \(\mathcal{A}_0^*\) such that \(\mathcal{A}_0^*\) is a \(\tau^*\)-submodel of \(\mathcal{A}^*\).

**Theorem 2.10** (Monster model). Let \(\mu\) be a cardinal. There is \(M \in \mathcal{K}\) such that:

1. **Universality:** \(M\) is \(\mu\)-universal, that is for all \(\mathcal{A} \in \mathcal{K}\), \(|\mathcal{A}| < \mu\), there is a \(\mathcal{K}\)-embedding \(f : \mathcal{A} \rightarrow M\).

\(\text{Here } \bar{b} \subseteq \bar{a} \text{ means that } lg(\bar{b}) \leq lg(\bar{a}) \text{ and the members of the tuple } \bar{b} \text{ are contained in the set of members of } \bar{a}, \text{i.e. when } \bar{b} = (b_0, ..., b_k) \text{ and } \bar{a} = (a_0, ..., a_n), \{b_0, ..., b_k\} \subseteq \{a_0, ..., a_n\}.\)
(2) \textit{\(K\)-homogeneity:} For all \(\mathcal{A} \preceq_K \mathcal{M}\) such that \(|\mathcal{A}| < \mu\) and mappings \(f : \mathcal{A} \to \mathcal{M}\) such that for all finite tuples \(\bar{a} \in \mathcal{A}\)
\[\text{tp}_{\mathcal{M}}^g(\bar{a}/\emptyset) = \text{tp}_{\mathcal{M}}^g(f(\bar{a})/\emptyset),\]
there is \(g \in \text{Aut}(\mathcal{M})\) extending \(f\).

From the property (2) it follows that for any \(\mathcal{A} \preceq_K \mathcal{M}\) a \(K\)-embedding \(f : \mathcal{A} \to \mathcal{M}\) extends to an automorphism of \(\mathcal{M}\), which is the standard definition of \(K\)-homogeneity. Finite character gives the above formulation, since we have that when \(\mathcal{A} \preceq_K \mathcal{M}\), \(f : \mathcal{A} \to \mathcal{M}\) is a \(K\)-embedding if and only if it is ‘type preserving’ in the sense of Theorem 2.10(2).

From now on we assume that we are in the monster model \(\mathcal{M}\), that is, each set is a subset of the monster model and each model is a \(K\)-elementary submodel of the monster model.

We know by Lemma 2.9 that \(\mathcal{M} = \mathcal{M}^* \upharpoonright \tau\) for some \(\mathcal{M}^*\) in the extended language \(\tau^*\), but we are not able to prove that \(\mathcal{M}^*\) would have the same homogeneity properties as in [6]. For any model \(N\) of \(\tau^*\) and a set \(A \subset N\) denote by \(SH^N(A)\) the ‘Skolem hull’ of \(A\) in \(N\), i.e. the closure of \(A\) under the functions of \(\tau^*\). For the monster model we abbreviate that \(SH^{\mathcal{M}^*}(A) = SH(A)\). By Lemma 2.8, for each subset \(A \subset \mathcal{M}\), always \(SH(A) \upharpoonright \tau \preceq_K \mathcal{M}\). Also Lemma 2.9 gives that for any particular model \(\mathcal{A} \preceq_K \mathcal{M}\) we may define \(\mathcal{M}^*\) so that \(SH(\mathcal{A}) = \mathcal{A}\). We will take advantage of this in Lemma 2.26. Thus we remark that although we fix a monster model \(\mathcal{M}\), there is no reason to fix any particular extension \(\mathcal{M}^* \in K^*\).

We recall the definitions of Galois type and weak type from [6]. Galois type is the usual notion, except that we define it over arbitrary sets, not only models. Weak type has a built-in finite character.

\textbf{Definition 2.11 (Galois type).} We write \(\text{tp}^g(\bar{a}/A) = \text{tp}^g(\bar{b}/A)\) if there is \(f \in \text{Aut}(\mathcal{M}/A)\) such that \(f(\bar{a}) = \bar{b}\).

The Galois type over \(\emptyset\) agrees with the type \(\text{tp}^g(\bar{a}/\emptyset, \mathcal{M})\) of Definition 2.4.

\textbf{Definition 2.12 (Weak type).} We write \(\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{b}/A)\) if \(\text{tp}^g(\bar{a}/B) = \text{tp}^g(\bar{b}/B)\) for each finite \(B \subset A\).

Clearly weak type and Galois type agree over finite sets.

\subsection*{2.1. Indiscernible sequences.} In [6] we used the homogeneity of \(\mathcal{M}^*\) to find suitable indiscernible sequences. The following proposition does the same in this context. The letters EM in the proposition come from Ehrenfeucht and Mostowski. The notion of an Ehrenfeucht-Mostowski model in the context of AEC’s was introduced by Shelah, and these models are a standard tool in the research of AEC’s, see
Proposition 2.13. (Shelah) Let \((\bar{b}_i)_{i<H}\) be a sequence of distinct tuples, let \(A\) be a countable set and let \((I, <), |I| < \mu\), be a linear ordering. Then there is a sequence \((\bar{a}_i)_{i \in I}\) and for each suborder \(J \subset I\) a model \(EM(J \cup A) \in \mathbb{K}^*, EM(J \cup A) \models \tau \leq_K \mathcal{M}\), with the following properties

1. When \(J \subset J' \subset I\),
   (a) \(A \cup (\bar{a}_i)_{i \in J} \subset EM(J \cup A)\),
   (b) each element of \(EM(J \cup A)\) is of the form \(t(\bar{d})\), where \(t\) is a term of \(\tau^*\) and \(\bar{d} \in J \cup A\),
   (c) \(EM(J \cup A)\) is a \(\tau^*\)-submodel of \(EM(J' \cup A)\) and
   (d) \(|EM(J \cup A)| = |J| + \aleph_0\).
2. For every finite \(i_0 < \ldots < i_n\) there are \(j_0 < \ldots < j_n < H\) such that
   \[tp^g(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/A) = tp^g(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n}/A).\]
   Furthermore, there is a \(\tau^*\)-isomorphism
   \[f : EM(i_0, \ldots, i_n \cup A) \to SH(\bar{b}_{j_0}, \ldots, \bar{b}_{j_n} \cup A)\]
   over \(A\) mapping \(\bar{a}_{i_k}\) to \(\bar{b}_{j_k}\) for each \(0 \leq k \leq n\).
3. If \(J \subset I\) and \(f : J \to I\) is order-preserving,
   \[tp^g((\bar{a}_i)_{i \in J}/A) = tp^g((\bar{a}_{f(i)})_{i \in J}/A).\]
   Furthermore, there is a \(\tau^*\)-isomorphism
   \[F : EM(J \cup A) \to EM((f(J) \cup A)\]
   over \(A\) mapping \(\bar{a}_i\) to \(\bar{a}_{f(i)}\) for each \(i \in J\).
4. For any linear order \(I'\), \(|I'| < \mu\), extending \(I\) we can extend the sequence to \((\bar{a}_i)_{i \in I'}\) with coherent models \(EM(J \cup A)\), for suborders \(J \subset I'\), with property (3).

Proof. We may assume that all the tuples \(\bar{b}_i, i < H\), have the same length. For a shorter notation we write \(\bar{b}_i = b_i\) and \(\bar{a}_j = a_j\) for all \(i < H, j \in I\). We write \(tp^*(\bar{b}/A) = tp^*(\bar{c}/A)\), if

\[\mathcal{M}^*, (a)_{a \in A} \models \varphi(\bar{b}) \iff (\mathcal{M}^*, (a)_{a \in A} \models \varphi(\bar{c})\]

for all atomic formulas \(\varphi(x)\) in the vocabulary \(\tau^* \cup (a)_{a \in A}\). Using the Erdős-Rado theorem we can define, by induction on \(n < \omega\), sets \(I^n_\alpha \subset \mathcal{M}^{*\lg(b_\alpha)}, \alpha < (2^{\aleph_0})^+\) such that

(i) \(I^n_\alpha \subset (b_i)_{i < H}\).
(ii) \(|I^n_\alpha| \geq \beth_\alpha\).

(iii) When \(b_0, ..., b_{n-1} \in I^n_\alpha\) with increasing indexes and \(c_0, ..., c_{n-1} \in I^n_\beta\) with increasing indexes,

\[
\text{tp}^*(b_0, ..., b_{n-1}/A) = \text{tp}^*(c_0, ..., c_{n-1}/A).
\]

(iv) When \(b_0, ..., b_{n-1} \in I^n_\alpha\) with increasing indexes and \(c_0, ..., c_{n-1} \in I^m_\alpha\) with increasing indexes and \(m > n\),

\[
\text{tp}^*(b_0, ..., b_{n-1}/A) = \text{tp}^*(c_0, ..., c_{n-1}/A).
\]

After that we choose some \(\alpha < (2^{n_0})^+\) and \(a_0^n, ..., a_n^n \in I^n_\alpha\) with increasing indexes for all \(n < \omega\).

Let \(I'\) be any linear order extending \(I\). Let \((a)_{a \in A}\) and \((c_i)_{i \in I'}\) be new constants. Define a consistent theory

\[
T_{I'} = \{ \varphi(c_{j_0}, ..., c_{j_n}, \bar{a}) : (\mathfrak{M}^*, a)_{a \in A} \models \varphi(a_0^n, ..., a_n^n, \bar{a}), j_0, < ... <, j_n \in I', \bar{a} \in A, \\
\text{\varphi an atomic sentence of the vocabulary } \tau^* \cup (a)_{a \in A}\}.
\]

Theory \(T_{I'}\) has a model \(N\), where we denote \(A^N = (a^N)_{a \in A}\) and \(I^N = (c^N_i)_{i \in I'}\).

We want to show that \(SH^N(I^N \cup A^N)\) is in \(\mathbb{K}^*\). To show that, we claim

(a) for every subset \(J \subset I^N\), \(SH^N(J \cup A^N)\) is \(\in \mathbb{K}^*\) and

(b) when \(J_1 \subset J_2 \subset I^N\), \(SH^N(J_1 \cup A^N) \upharpoonright \tau \leq_{\mathbb{K}} SH^N(J_2 \cup A^N) \upharpoonright \tau\).

We prove the claim by induction on the size of the subset \(J\). When \(J\) is finite, \(SH^N(J \cup A^N)\) is isomorphic to \(SH^{MT}(a_0^n, ..., a_n^n \cup A^N)\) in \(\mathbb{K}^*\) for some finite \(n\). Then assume that the claim holds for cardinals strictly smaller than \(|J|\). We have that \(J = \bigcup_{i < |J|} J_i\), where \(|J_i| < |J|\) for all \(i < |J|\). Then also \(SH^N(J \cup A^N) = \bigcup_{i < |J|} SH^N(J_i \cup A^N) \in \mathbb{K}^*\). Item (b) follows from item (a) and Lemma 2.8.

Since \(SH^N(I^N \cup A^N) \upharpoonright \tau \in \mathbb{K}\), we may assume that it is a \(\mathbb{K}\)-elementary submodel of \(\mathfrak{M}\). There is an isomorphism from \(SH^N(A^N)\) to \(SH(A)\) respecting the functions of \(\tau^*\). Thus by \(\mathbb{K}\)-homogeneity of \(\mathfrak{M}\) we may assume that \(SH^N(A^N) = SH(A)\).

We claim that \((c_i^N)_{i \in I}\) is the sequence we want, and the models \(EM(J \cup A)\) can be defined as \(SH^N(J^N \cup A)\) for \(J \subset I\).

For any finite \(i_0 < ... < i_n \in I\), \(SH^N(c_{i_0}^N, ..., c_{i_n}^N \cup A)\) is \(\tau^*\)-isomorphic to \(SH(a_0^n, ..., a_n^n \cup A)\) over \(A\), where \(a_k^n = b_{i_k}\) for some \(i_0 < ... < i_n < H\). Since both \(SH^N(c_{i_0}^N, ..., c_{i_n}^N \cup A) \upharpoonright \tau \leq_{\mathbb{K}} \mathfrak{M}\) and \(SH(a_0^n, ..., a_n^n \cup A) \upharpoonright \tau \leq_{\mathbb{K}} \mathfrak{M}\), this shows that (2) holds.

Furthermore, for any \(J \subset I\) and order-preserving \(f : J \rightarrow I\), \(SH^N(J \cup A)\) is \(\tau^*\)-isomorphic to \(SH^N(f(J) \cup A)\) over \(SH^N(A)\), with an isomorphism mapping \(c_i^N\) to \(c_{f(i)}^N\) for each \(i \in J\). Thus (3) holds.
We have that if $N'$ is a model of $T_J$, $N''$ is a model of $T_{J'}$ and $J'$ is a linear order extending $J$, then $SH^N(J^A \cup A^N)$ is $\tau^\ast$-isomorphic to $SH^{N''}(J^{N'} \cup A^{N''})$. Again from universality and $\mathbb{K}$-homogeneity of $\mathfrak{M}$ we get that $SH^N((c_i^N)_{i \in I} \cup A) \upharpoonright \tau$ can be extended to a model $SH^N((c_i^N)_{i \in I'} \cup A) \upharpoonright \tau \preccurlyeq \mathfrak{M}$, where $N'$ is a model of the theory $T_{J'}$ for arbitrary linear $I'$ extending $I$. This shows that (4) holds. □

When $\bar{a}$ is an $n$-tuple in $EF(I \cup A)$ as in the previous theorem, and $I \subset \mathfrak{M}$ a linear order, we have that

$$\bar{a} = (\bar{t}(i_1, ..., i_{k_0}, \bar{a}_0), ..., \bar{t}_{n-1}(i_{0}^{n-1}, ..., i_{k_{n-1}}^{n-1}, \bar{a}_{n-1})), $$

for some $\{i_0^1, ..., i_{k_0}^1, ..., i_{0}^{n-1}, ..., i_{k_{n-1}}^{n-1}\} \subset I$ and $\{\bar{a}_0, ..., \bar{a}_{n-1}\} \subset A$. We want to use a shorter notation and write

$$\bar{a} = \bar{t}(i_0, ..., i_k, \bar{a}'),$$

where $(i_0, ..., i_k) = (i_0^1, ..., i_{k_0}^1, ..., i_0^{n-1}, ..., i_{k_{n-1}}^{n-1})$ and $\bar{a}' = (\bar{a}_0, ..., \bar{a}_n)$. We say that $\bar{a}$ is generated by the sequence $\bar{t}$ of terms from $(i_0, ..., i_k)$ and $\bar{a}'$.

We recall the definition of a strongly $A$-indiscernible sequence from [6].

**Definition 2.14** (Strong indiscernibility). We say that a sequence $(\bar{a}_i)_{i < \alpha}$ of tuples is strongly indiscernible over $A$, or strongly $A$-indiscernible, if for every ordinal $\lambda \geq \alpha$ there is a sequence $(\bar{a}_i)_{i < \lambda}$ extending $(\bar{a}_i)_{i < \alpha}$ such that for any order-preserving partial $f : \lambda \rightarrow \lambda$, there is $F \in \text{Aut}(\mathfrak{M}/A)$ such that $F(\bar{a}_i) = \bar{a}_{f(i)}$ for all $i \in \text{dom}(f)$.

We remark that a constant sequence is strongly $A$-indiscernible for any $A$. By Proposition 2.13 we easily see the following.

**Proposition 2.15.** Let $(\bar{b}_i)_{i < H}$ be a sequence of distinct tuples and let $A$ be a countable set. There is a strongly $A$-indiscernible sequence $(\bar{a})_{i < \omega}$ such that for each $n < \omega$ there are $i_0 < ... < i_n < H$ such that

$$\text{tp}^Q(\bar{a}_0, ..., \bar{a}_n/A) = \text{tp}^Q(\bar{b}_{i_0}, ..., \bar{b}_{i_n}/A).$$

In the following we define a technical concept called a tidy sequence. The concept will help us to form indiscernible sequences of tuples, where each element of a tuple is contained in another indiscernible sequence.

**Definition 2.16** (Tidy sequence). Let $i_0^\alpha < ... < i_n^\alpha < \kappa$ for each $\alpha < \omega_1$, where $\kappa$ is a cardinal. We say that the sequence $(i_0^\alpha, ..., i_n^\alpha)_{\alpha < \omega_1}$ is tidy, if for each $0 \leq k \leq n$ one of the following holds.

1. The index at $k$ is constant, that is, $i_k^\alpha = \beta < \kappa$ is fixed for each $\alpha < \omega_1$.
2. The index at $k$ is included in some $(m+1)$-block, that is, $k \in \{p, p+1, ..., p+(m-1)\}$ such that
We note that, in the previous definition, the index \( k \) is said to be in a 1-block, if whenever \( \alpha < \beta \) we have \( i_k^\beta < i_k^\alpha \), all indexes \( i_{k-1}^\beta \) are smaller than \( i_k^0 \) and all indexes \( i_{k+1}^\beta \) are greater or equal to \( \sup\{i_k^\alpha : \alpha < \omega_1\} \).

**Lemma 2.17.** Assume that \( i_0^\alpha < \ldots < i_n^\alpha < \kappa \) for all \( \alpha < \omega_1 \), where \( \kappa \) is a cardinal. There is an uncountable subsequence \((i_0^{\alpha_j}, \ldots, i_{\alpha_j}^{\alpha_j})_{j<\omega_1}\) which is tidy.

**Proof.** We first claim that whenever \((i_\alpha)_{\alpha<\omega_1}\) are different indexes such that \( i_\alpha < \kappa \), there is a subsequence \((i_{\alpha_j})_{j<\omega_1}\) such that \( j < j' < \omega_1 \) implies \( i_{\alpha_j} < i_{\alpha_j'} \). We prove the claim by choosing for \( j < \omega_1 \) such \( \alpha_j \) that
\[
i_{\alpha_j} = \min\{i_\alpha < \kappa : \alpha > \sup\{\alpha_{j'} : j' < j\} \text{ and } i_\alpha > \sup\{i_{\alpha_{j'}} : j' < j\}\}.
\]

Now since for any countable \( j \) there are still uncountably many \( i_\alpha \) to choose from, the above set is nonempty.

We prove the lemma by induction on \( n \). First the case when \( n = 0 \). Either there are uncountable many same indexes \( i_0^\alpha \) or if not, there are uncountably many different. In the first case we find a constant subsequence, and in the second case we find a 1-block by the previous claim.

Then assume that we have shown the lemma for \( n \), and need to find a tidy subsequence of \((i_0^\alpha, \ldots, i_{n+1}^\alpha)_{\alpha<\omega_1}\). By induction, we may assume that \((i_0^\alpha, \ldots, i_n^\alpha)_{\alpha<\omega_1}\) is tidy. First we look at the case where there are uncountably many \( \alpha < \omega_1 \) such that \( i_{n+1}^\alpha \geq \sup\{i_n^\alpha : \alpha < \omega_1\} \). If so, we do as in the case \( n = 0 \). Otherwise there must be a subsequence \((\alpha_k)_{k<\omega_1}\) of different indexes such that
\[
\sup\{i_k^{\alpha_k} : k < \omega_1\} = \sup\{i_n^\alpha : \alpha < \omega_1\}.
\]

By renumering the sequence we write \( \alpha_k = \gamma < \omega_1 \). By the previous claim we may assume that \( \gamma' < \gamma < \omega_1 \) implies \( i_{n+1}^{\gamma'} < i_{n+1}^\gamma \). Since the index at \( n \) can’t be a constant, there is \( m < \omega \) and an \((m+1)\)-block for indexes \( p, \ldots, p+m \) such that \( p+m = n \). But now we define the subsequence \((i_0^{\alpha_j}, \ldots, i_{n+1}^{\alpha_j})_{\beta<\omega_1}\) of \((i_0^{\gamma}, \ldots, i_{n+1}^{\gamma})_{\gamma<\omega_1}\) as follows. For each countable \( j \), let \( \alpha_j \) be such that
\[
i_p^{\alpha_j} = \min\{i_p^\gamma < \kappa : i_p^\gamma > i_{n+1}^{\alpha_{j'}} \text{ for each } j' < j\}.
\]

Again the minimum exists as a minimum of a non-empty subset of a well-order. Now \((i_0^{\alpha_j}, \ldots, i_{p+m}^{\alpha_j}, i_{n+1}^{\alpha_j})_{j<\omega_1}\) forms an \((m+2)\)-block and thus the sequence \((i_0^{\alpha_j}, \ldots, i_{n+1}^{\alpha_j})_{j<\omega_1}\) is tidy. \(\square\)
Lemma 2.18. Let $EM(\kappa \cup A)$ be as in Proposition 2.13, for a sequence $(\bar{a}_i)_{i<\kappa}$ and a countable set $A$. Let

$$\bar{b}_\alpha = \bar{f}(\bar{a}_{i_0}, ..., \bar{a}_{i_n}, \bar{a})_{\alpha<\omega_1}$$

be a sequence of tuples such that $\bar{f}$ is a sequence of terms of $\tau^*$, $\bar{a} \in A$ fixed and $i_0^\alpha < ... < i_n^\alpha < \kappa$ for each $\alpha < \omega_1$. Then there exists an uncountable subsequence $(\bar{b}_\beta)_{\beta<\omega_1}$ such that it is a strongly $A$-indiscernible sequence.

Proof. By Lemma 2.17 we find a subsequence $(i_0^\beta, ..., i_n^\beta)_{\beta<\omega_1}$ which is tidy. We want to show that the sequence $(\bar{b}_\beta)_{\beta<\omega_1}$ is strongly $A$-indiscernible.

Let $\lambda \geq \omega_1$ be an ordinal. We want to extend the sequence to a suitable sequence $(i_0^\beta, ..., i_n^\beta)_{\beta<\lambda}$. First by (4) of Proposition 2.13, we can extend $(\bar{a}_i)_{i<\kappa}$ to $(\bar{a}_i)_{i \in I}$, where $I$ is a $|\lambda|^+$-dense linear order extending $\kappa$.

For an index $k$ there are two different cases.

1. $i_k^\gamma = \beta$ is constant for all $\gamma < \omega_1$.
2. $(i_k^\gamma, ..., i_{k+m}^\gamma)_{\gamma<\omega_1}$ forms an $(m+1)$-block.

For the case in (1), take $i_k^\alpha = \beta$ for all $\alpha < \lambda$. In the case of (2), we use $|\lambda|^+$-density to extend the $(m+1)$-block to $(i_k^\gamma, ..., i_{k+m}^\gamma)_{\alpha<\lambda}$ such that for $\gamma' < \gamma < \lambda$ we have

$$i_k^\gamma < ... < i_{k+m}^\gamma < i_k^\gamma < ... < i_{k+m}^\gamma < \min \{i_{k+m+1}^\alpha : \alpha < \omega_1\}$$

Finally, let $f : \lambda \rightarrow \lambda$ be partial and order-preserving. We have that the mapping $i_k^\beta \mapsto i_k^{f(\beta)}$, for all $0 \leq k \leq n$ and $\beta \in \text{dom}(f)$, preserves the ordering of $I$. Hence by (3) of Proposition 2.13, there is a $\tau^*$-isomorphism

$$F : EM((i_0^\beta, ..., i_n^\beta)_{\beta \in \text{dom}(f)} \cup A) \rightarrow EM((i_0^\beta, ..., i_n^\beta)_{\beta \in \text{rng}(f)} \cup A)$$

fixing $A$ pointwise and mapping $(\bar{a}_{i_0^\beta}, ..., \bar{a}_{i_n^\beta})$ to $(\bar{a}_{i_{f(\beta)}}, ..., \bar{a}_{i_{f(\beta)}})$ for all $\beta \in \text{dom}(f)$. This mapping $F$ extends to an automorphism of $\mathfrak{M}$ mapping $\bar{b}_\beta$ to $\bar{b}_{f(\beta)}$ for all $\beta \in \text{dom}(f)$. \qed

2.2. Categoricity and stability. In [6], we had two different notions of saturation. A model $\mathcal{A}$ is weakly $\kappa$-saturated, if the following holds: Let $B \subset \mathcal{A}$, $|B| < \kappa$ and $\bar{b} \in \mathfrak{M}$. Then $\text{tp}^w(\bar{b}/B)$ is realized in $\mathcal{A}$. A model $\mathcal{A}$ is weakly saturated, if it is weakly $|\mathcal{A}|$-saturated. Similarly a model is $\kappa$-saturated or saturated, if the same holds for Galois types. Clearly weak $\aleph_0$-saturation is the same notion as $\aleph_0$-saturation, since over finite sets Galois types and weak types are the same. We also discussed $\kappa$-categoricity for a cardinal $\kappa$, that is, each model of size $\kappa$ being isomorphic. First we recall the notion of $\aleph_0$-stability.
Definition 2.19 (N\(_0\)-stability). We say that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is \(\aleph_0\)-stable if for all countable \(A\) and uncountable sequence \((\bar{a}_i)_{i<\omega_1}\), there are \(i < j < \omega_1\) such that \(tp^w(\bar{a}_i/A) = tp^w(\bar{a}_j/A)\).

Although our notion of \(\aleph_0\)-stability refers to weak types, we will not call it weak \(\aleph_0\)-stability. We recall an important theorem from [6], there called Theorem 3.12. This theorem implies that our notion of \(\aleph_0\)-stability implies also \(\aleph_0\)-stability respect to Galois types. The proof of this theorem is identical to the one in [6].

Theorem 2.20. Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be an \(\aleph_0\)-stable finitary AEC. Assume that \(\mathcal{A}\) is a countable model and \(tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A})\). Then there is \(f \in Aut(\mathcal{M}/\mathcal{A})\) such that \(f(\bar{a}) = \bar{b}\).

Also the following result is stated as Corollary 4.8. of [6], and we can again imitate the proof.

Lemma 2.21. Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be an \(\aleph_0\)-stable finitary abstract elementary class and suppose that \(\lambda \geq H\). Then there is an \(\aleph_1\)-saturated model of size \(\lambda\).

We recall a well-known fact about Galois types. The result can be found in [15], see [10] for an easy proof.

Lemma 2.22.

1. Let \(\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}, \mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}'\), \(|\mathcal{A}| < |\mathcal{B}'| \leq \kappa\) and \(\mathcal{B}\) be \(\kappa\)-saturated. Then there is an automorphism \(f \in Aut(\mathcal{M}/\mathcal{A})\) such that \(f(\mathcal{B}') \preceq_{\mathbb{K}} \mathcal{B}\).

2. Two saturated models \(\mathcal{B}_1, \mathcal{B}_2\) containing \(\mathcal{A}\), such that \(|\mathcal{A}| < |\mathcal{B}_1| = |\mathcal{B}_2|\), are isomorphic over \(\mathcal{A}\).

We have that \(\aleph_0\)-stability of the class \((\mathbb{K}, \preceq_{\mathbb{K}})\) implies that there are countable \(\aleph_0\)-saturated models. Since any two countable \(\aleph_0\)-saturated models are isomorphic, we gain from the previous theorem that any two \(\kappa\)-saturated models of size \(\kappa\) are isomorphic. Then, under \(\aleph_0\)-stability, if all models of size \(\kappa\) are \(\kappa\)-saturated, the class is categorical in \(\kappa\). This gives raise to the following concept of weak categoricity.

Definition 2.23 (Weak categoricity). We say that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is weakly \(\kappa\)-categorical if each model of size \(\kappa\) is weakly saturated.

Theorem 2.24. Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be a finitary abstract elementary class, which is weakly \(\kappa\)-categorical for some uncountable \(\kappa\). Then it is \(\aleph_0\)-stable.

Proof. Let \(A\) be a countable set. By Proposition 2.13 there is a model \(EM(\kappa \cup A)\) of size \(\kappa\) such that each element is of the form \(t(i_0, ..., i_n, \bar{a})\) for a term \(t\) of \(\tau^*\),
\{i_0, ..., i_n\} \subset \kappa \text{ and } \bar{a} \in A, \text{ and each partial order-preserving } f : \kappa \to \kappa \text{ induces an automorphism } F \in \text{Aut}(\mathcal{M}/A) \text{ mapping } i \text{ to } f(i) \text{ for each } i \in \text{dom}(f).

Since |EM(\kappa \cup A)| = \kappa, it is saturated. Each weak type over A is realized in SH(\kappa \cup A). We assume that (\bar{b}_i)_{i < \omega_1} are elements of EM(\kappa \cup A). To prove the theorem, we should find \(i < j < \omega_1\) such that tp^w(\bar{b}_i/A) = tp^w(\bar{b}_j/A).

Since \(\tau^*\) and \(\mathcal{A}\) are countable, we may assume that each \(\bar{b}_i\) is of the form 
\[\bar{b}_i = \bar{t}(j_{i_0}^i, ..., j_{i_n}^i, \bar{a})\]
for a fixed sequence t of terms of \(\tau^*\) and fixed \(\bar{a} \in \mathcal{A}\). Furthermore, we may assume that each \(\{j_0^i, ..., j_n^i\}\) have the same order in \(\kappa\). But then tp^w(\bar{b}_i/A) = tp^w(\bar{b}_j/A) for all \(i < j < \omega_1\). \hfill \Box

Shelah has shown that also \(\kappa\)-categoricity implies \(\aleph_0\)-stability. The proof is similar to the previous proof. We assume that there would be \((\bar{b}_i)_{i < \omega_1}\) with different type over \(A\). Both \(A\) and \((\bar{b}_i)_{i < \omega_1}\) are included in some model \(\mathcal{B}\) of size \(\kappa\). But \(\kappa\)-categoricity implies that \(\mathcal{B}\) is isomorphic to a model of type \(EM(\kappa) = EM(\kappa \cup \emptyset)\) as above. But any countable subset \(A'\) of \(EM(\kappa)\) is generated by a countable well-order \(I \subset \kappa\), and terms \(\bar{t}(\bar{a})\) and \(\bar{t}(\bar{b})\) have the same type over \(A'\) if and only if there is a partial order-preserving \(f : \kappa \to \kappa\) fixing \(I\) and mapping \(\bar{a}\) to \(\bar{b}\). Thus we can see that there can be only countably many different Galois types over any countable subset of \(EM(\kappa)\), a contradiction. By combining these two results we get the following.

**Lemma 2.25.** Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is a finitary AEC. Then \((\mathbb{K}, \leq_{\mathbb{K}})\) is weakly \(\aleph_1\)-categorical if and only if it is \(\aleph_1\)-categorical.

**Proof.** Both \(\aleph_1\)-categoricity and weak \(\aleph_1\)-categoricity imply \(\aleph_0\)-stability. By Theorem 2.20, a weakly \(\aleph_1\)-saturated model is also \(\aleph_1\)-saturated respect to Galois types. Let \(\mathcal{A}_1\) and \(\mathcal{B}_2\) be two saturated models of size \(\aleph_1\). By \(\aleph_0\)-stability we may assume that they have a common \(\aleph_0\)-saturated countable submodel, and then by Lemma 2.22 are isomorphic.

We prove the other direction. By \(\aleph_0\)-stability, we can build a (weakly) \(\aleph_1\)-saturated model as an increasing chain of countable models \(\mathcal{A}_i\), \(i < \omega_1\), such that each (weak) type over \(\mathcal{A}_i\) is realized in \(\mathcal{A}_{i+1}\). Then by \(\aleph_1\)-categoricity, each model of size \(\aleph_1\) is isomorphic to this model. \hfill \Box

We state another corollary of Proposition 2.12.

**Lemma 2.26.** Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is weakly categorical in an uncountable cardinal \(\kappa\). Then each model of size \(\mathbb{H}\) is \(\aleph_1\)-saturated.
Proof. Let $\mathcal{A} = (b_i)_{i < H}$ be a model of size $H$ and $A \subset \mathcal{A}$ a countable set. By Lemma 2.9 we may assume that $SH(\mathcal{A}) = \mathcal{A}$. Let $tp^w(\bar{a}/A)$ be a weak type over $A$. Let $EM(I \cup A)$, for each $I \subset \kappa$, be as in Proposition 2.13. Now $A \subset EM(\kappa \cup A)$ and $EM(\kappa \cup A)$ is $\aleph_1$-saturated by weak $\kappa$-categoricity.

The type $tp^w(\bar{a}/A)$ is realized in $EM(i_0, ..., i_n \cup A)$ for some finite $i_0 < ... < i_n < \kappa$. By condition (2) of Proposition 2.13 there are $j_0 < ... < j_n < H$ and a $\tau^*$-isomorphism

$$f : EM(i_0, ..., i_n \cup A) \to SH(b_{j_0}, ..., b_{j_n} \cup A),$$

fixing $A$ pointwise. Now $tp^w(\bar{a}/A)$ is realized in $SH(b_{j_0}, ..., b_{j_n} \cup A) \upharpoonright \tau \equiv_K \mathcal{A}$. □

2.3. Splitting. We recall the definitions of splitting and the notion of independence based on splitting. In [6], the homogeneity of the model $\mathfrak{M}^*$ was used to study the properties of this notion. We reprove those results concerning the extension property and symmetry.

Definition 2.27 (Splitting). We say that the weak type $tp^w(\bar{a}/A)$ splits over finite $B \subset A$ if there are $\bar{c}, \bar{d} \in A$ such that

$$tp^w(\bar{c}/B) = tp^w(\bar{d}/B) \text{ but }$$

$$tp^w(\bar{c}/B \cup \{\bar{a}\}) \neq tp^w(\bar{d}/B \cup \{\bar{a}\}).$$

We say that such $\bar{c}, \bar{d}$ witness the fact.

Definition 2.28 (Independence). We write $\bar{a} \downarrow^s_A B$ if $tp^w(\bar{a}/A \cup B)$ does not split over a finite subset of $A$.

As in Theorem 3.17 of [6], we can now prove the following properties of $\downarrow^s$ over $\aleph_0$-saturated models. Finite character of $(\mathbb{K}, \preceq_{\mathbb{K}})$ and $\aleph_0$-stability are needed in the proof of local character, and then local character is needed to prove properties (4)-(7).

Theorem 2.29. Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an $\aleph_0$-stable finitary AEC.

1. **Monotonicity** If $A \subset B \subset C \subset D$ and $\bar{a} \downarrow^s_A D$, then $\bar{a} \downarrow^s_B C$.
2. **Invariance** If $f$ is an automorphism of $\mathfrak{M}$, $\bar{a} \downarrow^s_A B$ if and only if $f(\bar{a}) \downarrow^s_{f(A)} f(B)$.
3. **Local character** For each model $\mathcal{A}$ and a finite sequence $\bar{a}$ there is finite $E \subset \mathcal{A}$ such that $\bar{a} \downarrow^s_E \mathcal{A}$. 
(4) **Countable extension** Let $\mathcal{A}$ be a countable $\aleph_0$-saturated model. Let $B$ be a countable set containing $\mathcal{A}$. For each $\bar{a}$ there is $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{A})$ such that $\bar{b} \downarrow A B$. Moreover, if $tp^w(\bar{a}/\mathcal{A})$ does not split over some finite subset $E$, then $tp^w(\bar{b}/B)$ does not split over the same finite set $E$.

(5) **Stationarity** Assume $\mathcal{A}$ is an $\aleph_0$-saturated model and $A \subset B$. If $tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow A B$ and $\bar{b} \downarrow A B$, then $tp^w(\bar{a}/B) = tp^w(\bar{b}/B)$.

(6) **Transitivity** Let $A \subset B \subset C$ and $B$ be an $\aleph_0$-saturated model. Then $\bar{a} \downarrow A C$ if and only if $\bar{a} \downarrow A B$ and $\bar{a} \downarrow B C$.

(7) **Finite character** Let $E$ be finite and $E \subset B$. Then $\bar{a} \downarrow E B$ if and only if $\bar{a} \downarrow E B_0$ for every finite $B_0 \subset B$.

The same holds if instead of $E$ we have an $\aleph_0$-saturated model $\mathcal{A}$.

With (4) of the previous theorem we can only find non-splitting extensions to types over countable sets. To prove symmetry, we need extensions to types over larger sets also. In [6], we showed that the general extension property is implied either by tameness\(^2\) or categoricity above the Hanf number. The latter proof used the homogeneous model $\mathcal{M}^*$, and we should reprove the result in this context.

**Definition 2.30 (Extension property).** We say that $(\mathcal{K}, \preceq_{\mathcal{K}})$ has the $\lambda$-extension property if the following holds:

Let $\mathcal{A}$ be an $\aleph_0$-saturated model and let $B$ contain $\mathcal{A}$, $|B| < \lambda$. Assume that $tp^w(\bar{a}/\mathcal{A})$ does not split over some finite subset $E$. Then there exists $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{A})$ such that $tp^w(\bar{b}/B)$ does not split over the set $E$.

We say that $(\mathcal{K}, \preceq_{\mathcal{K}})$ has the extension property if it has the $\lambda$-extension property for all cardinals $\lambda$.

The idea in the following proof is originally due to Shelah, and we formulated a version in Proposition 4.19. of [6], using the homogeneous extension $\mathcal{M}^*$. For completeness, we reprove the theorem in the context of Definition 2.6. We remark that only details have changed, and the general idea remains the same. We do not use finite character directly, only the properties of splitting. An easy proof in a more general context (with another notion of splitting) can be found in [1]. We will show that categoricity in some $\kappa \geq H$ implies the extension property. We remark that weak categoricity is not enough for this proof.

**Theorem 2.31 (Extension property).** Let $(\mathcal{K}, \preceq_{\mathcal{K}})$ be a finitary AEC, categorical in some $\kappa \geq H$. Then if $\mathcal{A}$ is an $\aleph_0$-saturated model, $\mathcal{A} \subset B$, $|B| \leq \kappa$ and $tp^w(\bar{a}/\mathcal{A})$ does not split over finite $E \subset \mathcal{A}$, there is $\bar{b}$ realizing $tp^w(\bar{a}/\mathcal{A})$ such that $tp^w(\bar{b}/B)$ does not split over $E$.

\(^2\)See definition in section 4.1.
Proof. Let $\mathcal{A}_0 \preceq_{K} \mathcal{A}$ be countable and $\aleph_0$-saturated such that $E \subseteq \mathcal{A}_0$. For every $\bar{b}$ realizing $\text{tp}^w(\bar{a}/\mathcal{A}_0)$ such that $\bar{b} \downarrow_{\mathcal{A}_0} B$ we get by stationarity that $\bar{b}$ also realizes $\text{tp}^w(\bar{a}/\mathcal{A})$. Thus we may assume that $\mathcal{A}$ is countable.

Let $(I, <)$ be a $\lambda^+$-dense linear order without endpoints extending the linear order $(Q \times \kappa, <)$. By Proposition 2.13 there are models $EM(Q \times \kappa) \subseteq EM(I) \in K^*$ for orders $(Q \times \kappa, <)$ and $(I, <)$ respectively such that $Q \times \kappa$ is a suborder of $I$ and any partial order-preserving $f : I \rightarrow I$ extends to an automorphism of $\mathcal{M}$ respecting the functions of $\tau^*$ on $EM(I)$.

The set $B \cup \{\bar{a}\}$ is included in $\mathcal{B}$ for some model $\mathcal{B}$ of size $\kappa$. From $\kappa$-categoricity we get that $\mathcal{B}$ and $EM(Q \times \kappa)$ are isomorphic. Thus we may assume that $B \cup \{\bar{a}\} \subseteq EM(Q \times \kappa)$. We have that $B \subseteq EM(K)$ for some $K \subseteq Q \times \kappa$ such that $|K| = \lambda$. We assumed that $\mathcal{A}$ is countable, and thus $\mathcal{A} \preceq_{K} EM(Q \times J)$ for some countable $J \subseteq \kappa$. We may also assume that $EM(Q \times J, <)$ is $\aleph_0$-saturated. By countable extension there is $\bar{a}' \in \mathcal{M}$ such that $\text{tp}^w(\bar{a}'/\mathcal{A}') = \text{tp}^w(\bar{a}/\mathcal{A})$ and $\text{tp}^w(\bar{a}'/EM(Q \times J))$ does not split over $E$. Since $EM(Q \times \kappa)$ is $\aleph_1$-saturated by categoricity and Lemma 2.21, we may assume that $\bar{a}'$ is in $EM(Q \times \kappa)$.

Let $i_0 < ... < i_{n-1} \subseteq Q \times \kappa$ and $\bar{t}$ be a finite sequence of terms of $\tau^*$ such that

$$\bar{a}' = \bar{t}(i_0, ..., i_{n-1}).$$

By $\lambda^+$-density of $I$ and density of $Q \times J$ we find $j_0 < ... < j_{n-1} \subseteq I$ such that

1. if $i_k \subseteq Q \times J$, then $j_k = i_k$,
2. $i_k < j$ if and only if $j_k < j$ for each $j \in Q \times J$,
3. if $i_k \notin Q \times J$, then $j_k \notin (Q \times J) \cup K$,
4. if there is $k \in K \setminus (Q \times J)$ such that $j_k < k < j_{k+1}$, then there are infinitely many $j \in Q \times J$ such that $j_k < j < j_{k+1}$,
5. if there is $k \in K$ between some $j_k$ and $j \in Q \times J$, then there are infinitely many $i \in Q \times J$ in that same interval,
6. if there are $k \in K$ such that $k < j_0$, then there is infinitely many such $j \in Q \times J$ and similarly for $k > j_{n-1}$.

Finally let $\bar{b} = \bar{t}(j_0, ..., j_{n-1})$. Now for every finite $J_0 \subseteq Q \times J$ there is order-preserving $f$ mapping $j_k$ to $i_k$ for every $0 \leq k < n$ such that $f \upharpoonright J_0 = \text{Id}_{J_0}$. Also for every finite $K_0 \subseteq K \setminus J_0$ we can extend this mapping such that it maps $K_0$ to $Q \times J$. We get that $\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})$ and $\text{tp}^w(\bar{b}/B)$ does not split over $E$. $\square$

The homogeneity of $\mathcal{M}^*$ was used in Theorem 4.3 and Proposition 4.4 of [6]. In the following we give proofs to those theorems in the present context. First we recall the following lemma (Lemma 4.2 of [6]), whose proof did not use $\mathcal{M}^*$.
Lemma 2.32. Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is an \(\aleph_0\)-stable finitary AEC and satisfies the H-extension property. Assume that \(\mathcal{A}\) is a countable \(\aleph_0\)-saturated model, \(\bar{a} \downarrow_{\mathcal{A}} b\) and \(\bar{b} \not\subseteq_{\mathcal{A}} \bar{a}\). Then there exists a sequence \((\bar{a}_i, \bar{b}_i)_{i < H}\) of length \(H\) such that \(\bar{b}_i \downarrow_{\mathcal{A}} \bar{a}_j\) if and only if \(i > j\).

Theorem 2.33 (Symmetry). Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is an \(\aleph_0\)-stable finitary AEC with H-extension property. Let \(\mathcal{A}\) be an \(\aleph_0\)-saturated model. If \(\bar{a} \downarrow_{\mathcal{A}} b\), then \(\bar{b} \downarrow_{\mathcal{A}} \bar{a}\).

Proof. We assume the contrary. Let \(\bar{a}\) and \(\bar{b}\) be such that \(\bar{a} \downarrow_{\mathcal{A}} b\) and \(\bar{b} \not\subseteq_{\mathcal{A}} \bar{a}\). By monotonicity and local character there is countable and \(\aleph_0\)-saturated \(\mathcal{A}_0 \lesssim_{\mathbb{K}} \mathcal{A}\) such that \(\bar{a} \downarrow_{\mathcal{A}_0} \mathcal{A} \cup \bar{b}\). Then by transitivity and monotonicity again, \(\bar{a} \downarrow_{\mathcal{A}_0} \bar{b}\) and \(\bar{b} \not\subseteq_{\mathcal{A}_0} \bar{a}\). Thus we may assume that \(\mathcal{A}\) is countable. Then we get from Lemma 2.32 a sequence \((\bar{a}_i, \bar{b}_i)_{i < H}\) such that

\[
\bar{b}_i \downarrow_{\mathcal{A}} \bar{a}_j\quad\text{if and only if}\quad i > j.
\]

Furthermore we use Proposition 2.13 to get a sequence \((\bar{c}_i, \bar{d}_i)_{i \in (\mathbb{R}, <')}\) such that for all \(i < j \in (\mathbb{R}, <')\) there are \(i' < j' < H\) such that

\[
\text{tp}^g(\bar{c}_i, \bar{d}_j/\mathcal{A}) = \text{tp}^g(\bar{a}_{i'}, \bar{b}_{j'}/\mathcal{A}),
\]

and thus

\[
\bar{d}_i \downarrow_{\mathcal{A}} \bar{c}_j\quad\text{if and only if}\quad j < i'.
\]

When we denote \(B = \mathcal{A} \cup \{(\bar{c}_i, \bar{d}_i) : i \in \mathbb{Q}\}\), \(B\) is countable and if \(i, j \in \mathbb{R}\) and \(i \neq j\), tuples \((\bar{c}_i, \bar{d}_i)\) and \((\bar{c}_j, \bar{d}_j)\) have different weak type over \(B\). This contradicts the \(\aleph_0\)-stability assumption. \(\square\)

The following proposition shows that H-extension property implies the extension property.

Proposition 2.34. Assume that \((\mathbb{K}, \leq_{\mathbb{K}})\) is an \(\aleph_0\)-stable finitary AEC with the H-extension property. Assume that \(\mathcal{A}\) is \(\aleph_0\)-saturated and \(\text{tp}^w(\bar{a}/\mathcal{A})\) does not split over some finite subset \(E\). Let \(\mathcal{A} \subseteq B\). Then there is \(b\) realizing \(\text{tp}^w(\bar{a}/\mathcal{A})\) such that \(\text{tp}^w(b/B)\) does not split over the set \(E\).

Proof. Let \(\mathcal{A}_0 \lesssim_{\mathbb{K}} \mathcal{A}\) be countable and \(\aleph_0\)-saturated such that \(E \subseteq \mathcal{A}_0\). For every \(b\) realizing \(\text{tp}^w(\bar{a}/\mathcal{A}_0)\) such that \(b \downarrow_{\mathcal{A}_0} B\) we get by stationarity that \(\bar{b}\) also realizes \(\text{tp}^w(\bar{a}/\mathcal{A})\). Thus we may assume that \(\mathcal{A}\) is countable. Denote \(|B|^+ = \lambda\).

Using H-extension property, we define \(\bar{a}_i\) for \(i < H\) such that

1. \(\bar{a}_i\) realizes \(\text{tp}^w(\bar{a}/\mathcal{A})\) for all \(i < H\),
2. \(\bar{a}_i \downarrow_{\mathcal{A}} SH(\bigcup_{j < i} \bar{a}_j \cup \mathcal{A})\).

By Proposition 2.13 we can find a sequence \(\bar{b}_j\), \(j < \lambda\), and models \(EM(J \cup \mathcal{A}) \in \mathbb{K}^*\), \(EM(J \cup \mathcal{A}) \uparrow \tau \lesssim_{\mathbb{K}} \mathfrak{M}\) for each \(J \subseteq \lambda\) with the following properties
(1) When \( I \subset J \subset \lambda \), \( \mathcal{A} \cup (\bar{a}_i)_{i \in I} \subset EM(I \cup \mathcal{A}) \subset EM(J \cup \mathcal{A}) \).

(2) For every finite \( I \subset \lambda \) there is finite \( J \subset H \) and an isomorphism
\[
f : SH((\bar{a}_j)_{j \in J} \cup \mathcal{A}) \to EM((I \cup \mathcal{A})
\]
over \( \mathcal{A} \) such that \( \bar{a}_j \)'s get mapped to \( \bar{b}_i \)'s order-preservingly.

First we claim the following:

(2.1) For each \( \bar{c} \in A \) there is finite \( X \subset \lambda \) such that \( \bar{c} \downarrow_A \bar{b}_i \) for each \( i \in \lambda \setminus X \).

If not, then there are \( \bar{c} \in A \) and \( i_0 < \ldots < i_k < \ldots < \lambda , k < \omega \), such that for all \( k < \omega \), \( \bar{c} \not\downarrow_{\mathcal{A}} \bar{b}_{i_k} \). Denote \( B_k = EM(i_0, \ldots, i_{k-1} \cup \mathcal{A}) \). By (2) and monotonicity,
\[
\bar{b}_{i_k} \downarrow_{\mathcal{A}} B_k \text{ for each } k < \omega .
\]
By symmetry, \( \bar{b}_{i_k} \not\downarrow_{\mathcal{A}} \bar{c} \), and by transitivity and symmetry again, \( \bar{c} \not\downarrow_{\mathcal{A}} \bar{b}_{i_k} \) for all \( k < \omega \). We get an increasing chain of models \( B_k \) such that \( \bar{c} \not\downarrow_{\mathcal{A}} B_{k+1} \) for all \( k < \omega \), a contradiction with local character of splitting. This proves (2.1).

Since \( \lambda > |A| \), there is \( i < \lambda \) such that \( \bar{c} \downarrow_{A} \bar{b}_i \) for all \( \bar{c} \in A \). But now we can take \( \bar{b} = \bar{b}_i \) by symmetry and finite character of splitting. \( \square \)

After we have shown symmetry, we can prove the following as Theorem 4.9 in [6].

**Theorem 2.35.** Assume that \( (\mathbb{K}, \preceq_\mathbb{K}) \) is a finitary AEC, stable in \( \aleph_0 \) and has the extension property. Then there is a weakly saturated model in every infinite cardinal \( \lambda \).

The previous theorem shows, that for any infinite \( \kappa \) and \( (\mathbb{K}, \preceq_\mathbb{K}) \) a finitary AEC with the extension property, \( \kappa \)-categoricity implies weak \( \kappa \)-categoricity.

### 2.4. Primary models.

We will define a concept of a primary model \( \mathcal{A}[\bar{a}] \), which is a constructible model over a countable model \( \mathcal{A} \) and a finite tuple \( \bar{a} \). A similar construction was used in [6] in the proof of Theorem 3.12, to show that weak types and Galois types agree over countable models. Under simplicity, we will define a concept of an \( f \)-primary model in section 4, which is constructible over \( \mathcal{A} \cup B \), where \( B \) is an arbitrary set. Throughout this section we will assume that \( (\mathbb{K}, \preceq_\mathbb{K}) \) is an \( \aleph_0 \)-stable finitary AEC with the extension property.

We recall from [6] the definition of a type being weakly isolated.

**Definition 2.36 (Weakly isolated type).** We say that a type \( tp^w(\bar{b}/\mathcal{A} \cup \bar{a}) \) is weakly isolated over finite \( A' \cup \bar{a} \), if whenever \( \bar{d} \) realizes \( tp^w(\bar{b}/A' \cup \bar{a}) \), then \( \bar{d} \) realizes \( tp^w(\bar{b}/\mathcal{A} \cup \bar{a}) \).

As in [6], we can use weak \( \aleph_0 \)-stability to show the following.
Lemma 2.37. Assume $\mathcal{A}$ is a countable model, $A$ a finite subset of $\mathcal{A}$ and $\bar{a}$ a finite tuple. Then for each $\bar{b}$ there are $\bar{c}$ and a finite $A' \subset \mathcal{A}$ such that

i) $\bar{c}$ realizes $tp^w(\bar{b}/A \cup \bar{a})$ and
ii) $tp^w(\bar{c}/\mathcal{A} \cup \bar{a})$ is weakly isolated over $A' \cup \bar{a}$.

In Lemma 3.8 of [6] we studied a countable set $A$ having the so called $\aleph_0$-saturation property, i.e. for every finite $A' \subset A$ and $\bar{a}$, the type $tp^w(\bar{a}/A')$ is realized in $A$. One consequence of finite character was that a countable set $A$ having this property is a $\preceq_{\mathcal{K}}$-substructure of the monster model i.e. a model. We can easily see that this generalizes to larger sets also, since an uncountable set with $\aleph_0$-saturation property can always be written as an increasing union of smaller sets with this property. Thus we write the fact as follows.

Proposition 2.38. Assume that $A$ is a set with the following property: For any finite $A' \subset A$ and $\bar{a}$, the type $tp^w(\bar{a}/A')$ is realized in $A$. One consequence of finite character was that a countable set $A$ having this property is a $\preceq_{\mathcal{K}}$-substructure of the monster model i.e. a model. We can easily see that this generalizes to larger sets also, since an uncountable set with $\aleph_0$-saturation property can always be written as an increasing union of smaller sets with this property. Thus we write the fact as follows.

We say that a sequence $(\bar{b}_i)_{i<\omega}$ of tuples is increasing if $j < i$ and $lg(\bar{b}_i) = n$ imply that $\bar{b}_i \upharpoonright n = \bar{b}_j$.

Definition 2.39 (Primary model). Assume that $\mathcal{A} = \{a_n : n < \omega\}$ is a countable model and $\bar{a}$ a finite tuple. Let $(\bar{b}_n)_{n<\omega}$, be increasing and $A_n$, $n < \omega$, finite such that:

1) $\bar{b}_0 = \bar{a}$ and $tp^w(\bar{a}/\mathcal{A})$ does not split over the finite subset $A_0$.
2) $a_n \cup A_n \subset A_{n+1} \subset \mathcal{A}$.
3) $tp^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{a})$ is weakly isolated over $A_n \cup \bar{a}$.
4) $tp^w(\bar{b}_{n+1}/\mathcal{A})$ does not split over $A_n$.
5) $\mathcal{A} \cup \bigcup_{n<\omega} \bar{b}_n$ is an $\aleph_0$-saturated model.

We call this model $\mathcal{A} \cup \bigcup_{n<\omega} \bar{b}_n$ the primary model $\mathcal{A}[\bar{a}]$.

We remark that since both $\mathcal{A}$ and $\mathcal{A}[\bar{a}]$ are $\mathcal{K}$-submodels of $\mathcal{M}$ and $\mathcal{A} \subset \mathcal{A}[\bar{a}]$, also $\mathcal{A} \preceq_{\mathcal{K}} \mathcal{A}[\bar{a}]$.

Lemma 2.40. Assume that $\mathcal{A}$ is countable. Then there exists a model $\mathcal{A}[\bar{a}]$ with properties (1) - (5) of Definition 2.39. If also $\mathcal{A} \preceq_{\mathcal{K}} \mathcal{B}$, $\bar{a} \in \mathcal{B}$ and $\mathcal{B}$ is $\aleph_0$-saturated, we can take $\mathcal{A}[\bar{a}] \preceq_{\mathcal{K}} \mathcal{B}$.

Proof. Let $\mathcal{A} = \{a_n : n < \omega\}$. By induction, we construct an increasing sequence of tuples $\bar{b}_n \in \mathcal{B}$, and finite sets $A_n \subset A$ $n < \omega$, such that they satisfy properties (1)- (5) of Definition 2.39.

For $n = 0$, choose $\bar{b}_0$ and $A_0$ as in (1), using local character of splitting. Assume that $\bar{b}_j \in \mathcal{B}$ and $A_j \subset A$ have been constructed for $j \leq n$. By $\aleph_0$-stability, we
can find \( \{ c^j_i : i < \omega, j \leq n \} \) realizing all the Galois types over \( A_j \cup b_j \), for \( j \leq n \).
Let \( a_n = (c^j_i)_{i,j \leq n} \). By Lemma 2.37 there exists \( A_{n+1} \) finite with \( A_n \subseteq A_{n+1} \subseteq \mathcal{A} \) and there exists \( \bar{d}^n \) realizing \( \text{tp}^w(\bar{b}^n \bar{d}^n/A_n) \) such that \( \text{tp}^w(\bar{b}' \bar{d}'/\mathcal{A} \cup \bar{a}) \) is weakly isolated over \( A_{n+1} \cup \bar{a} \). Since \( \text{tp}^w(\bar{b}' A_{n+1} \cup \bar{a}) = \text{tp}^w(b_n/A_{n+1} \cup \bar{a}) \) by induction hypothesis, we may assume that \( \bar{b}' = \bar{b}_n \). Then since \( \mathcal{B} \) is \( \aleph_0 \)-saturated and \( A_{n+1} \cup \bar{a} \cup \bar{b}_n \) is in \( \mathcal{B} \), we may also assume that \( \bar{d}' \) is in \( \mathcal{B} \). Let \( \bar{b}_{n+1} = \bar{b}_n \bar{d}' \in \mathcal{B} \).

By local character there is finite \( E \subseteq \mathcal{A} \) such that \( \text{tp}^w(\bar{b}_{n+1}/\mathcal{A}) \) does not split over \( E \). We take \( A_{n+1} = A_{n+1}' \cup E \cup a_n \). Then (1), (2), (3) are satisfied. Finite character is used to show (4) in the form of Lemma 2.38. Let \( c \in \mathfrak{M} \) and \( B \subseteq \mathcal{A} \cup \bigcup_{n<\omega} \bar{b}_n \) finite. By (2), \( \mathcal{A} = \bigcup_{n<\omega} A_n \), so there exists \( n < \omega \) such that \( B \subseteq A_n \cup \bar{b}_n \). Then \( \text{tp}^w(c/A_n \cup \bar{b}_n) \) is realized by some \( c^n_i \), and hence belongs to \( \bar{b}_{j+1} \) for some \( j \). We are done with the construction.

**Definition 2.41 (Domination).** We say that a set \( A \) dominates a model \( \mathcal{B} \) over a model \( \mathcal{A} \), if for every tuple \( \bar{c} \),

\[
\bar{c} \downarrow_{\mathcal{A}} A \iff \bar{c} \downarrow_{\mathcal{A}} \mathcal{B}.
\]

In the following lemma we see that \( \bar{a} \) dominates the primary model \( \mathcal{A}[\bar{a}] \) over \( \mathcal{A} \), when \( \mathcal{A} \) is \( \aleph_0 \)-saturated.

**Lemma 2.42.** Let \( \mathcal{A} \) be a countable \( \aleph_0 \)-saturated model and \( \bar{c}, \bar{a} \) finite. Then

\[
\bar{c} \downarrow_{\mathcal{A}}^* \bar{a} \text{ if and only if } \bar{c} \downarrow_{\mathcal{A}}^* \mathcal{A}[\bar{a}]\]

**Proof.** By monotonicity, the other direction is clear. We assume to the contrary, that \( \bar{c} \downarrow_{\mathcal{A}}^* \bar{a} \) and \( \bar{c} \not\downarrow_{\mathcal{A}}^* \mathcal{A}[\bar{a}] \). Let \( \mathcal{A}[\bar{a}] = \mathcal{A} \cup \bigcup_{i<\omega} \bar{b}_i \) be as in the definition of a primary model, where \( \bar{b}_i \) are increasing and \( \bar{b}_0 = \bar{a} \). Thus, by finite character of splitting, there is \( n \) such that \( \bar{c} \not\downarrow_{\mathcal{A}}^* \bar{b}_{n+1} \). Then, by symmetry, \( \bar{b}_{n+1} \not\downarrow_{\mathcal{A}}^* \bar{c} \) and thus \( \text{tp}^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{c}) \) splits over any finite \( E \subseteq \mathcal{A} \).

Let \( A_n \subseteq \mathcal{A} \) be the finite set such that \( \bar{b}_n \downarrow_{A_n} \mathcal{A} \) and \( \text{tp}^w(\bar{b}_{n+1}/\mathcal{A}) \) is weakly isolated over \( A_n \). We may find a finite \( B \) such that \( A_n \subseteq B \subseteq \mathcal{A} \) and \( \text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c}) \) splits over \( A_n \). By assumption and symmetry, \( \bar{a} \downarrow_{\mathcal{A}} \bar{c} \), and since \( \bar{a} \downarrow_{A_n} \mathcal{A} \), we get by transitivity and monotonicity that \( \bar{a} \downarrow_{B} \mathcal{A} \cup \bar{c} \).

Since \( \mathcal{A} \) is \( \aleph_0 \)-saturated, there is \( \bar{c}' \in \mathcal{A} \) such that \( \text{tp}^w(\bar{c}'/B) = \text{tp}^w(\bar{c}/B) \). But \( \text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{c}) \) does not split over \( B \), and thus we get an automorphism \( \phi \in \text{Aut}(\mathfrak{M}/B \cup \bar{a}) \) such that \( \phi(\bar{c}') = \bar{c}' \). By invariance, \( \text{tp}^w(\phi(\bar{b}_{n+1})/B \cup \bar{c}') \) splits over \( A_n \). On the other hand \( \text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c}') \) does not split over \( A_n \), and thus

\[
\text{tp}^w(\bar{b}_{n+1}/B \cup \bar{c}') \neq \text{tp}^w(\phi(\bar{b}_{n+1})/B \cup \bar{c}')\]

But \( \text{tp}^w(\bar{b}_{n+1}/A_n \cup \bar{a}) = \text{tp}^w(\phi(\bar{b}_{n+1})/A_n \cup \bar{a}) \), and this is a contradiction, since \( \text{tp}^w(\bar{b}_{n+1}/\mathcal{A} \cup \bar{a}) \) was weakly isolated over \( A_n \cup \bar{a} \).
In [6], we defined a concept of $U$-rank. With primary models we can show that if both $U(\bar{a}/\mathcal{A})$ and $U(\bar{b}/\mathcal{A})$ are finite for a countable $\aleph_0$-saturated model $\mathcal{A}$, then
\[(2.2) \quad U(\bar{a}\bar{b}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}) + U(\bar{b}/\mathcal{A}).\]
In the case of [6] we proved that one sufficient condition for simplicity for an $\aleph_0$-stable finitary AEC with extension property was the class having a finite $U$-rank. We defined that $(\mathcal{K}, \preceq_{\mathcal{K}})$ has finite $U$-rank, if for each finite sequence $\bar{a}$,
\[
\sup\{U(\bar{a}/\mathcal{A}) : \mathcal{A} \in \mathcal{K} \text{ countable and } \aleph_0\text{-saturated}\} < \aleph_0.
\]
We also mentioned that it is enough to study, instead of arbitrary finite tuples $\bar{a}$, only singletons $a$. The equation (2.2) shows this claim. As we will see in the following sections, we can develop the same theory of simplicity as in [6] for the class with assumptions of Definition 2.6.

We recall the definition of $U$-rank.

**Definition 2.43.** Let $\mathcal{A}$ be countable and $\aleph_0$-saturated model. Define $U$-rank of $\bar{a}$ over $\mathcal{A}$, $U(\bar{a}/\mathcal{A})$, by induction:

1. Always $U(\bar{a}/\mathcal{A}) \geq 0$.
2. $U(\bar{a}/\mathcal{A}) \geq \beta + 1$ iff there is countable $\aleph_0$-saturated model $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$, $U(\bar{a}/\mathcal{B}) \geq \beta$ and $\bar{a} \not\models_{\mathcal{B}} \mathcal{A}$

For a countable $\aleph_0$-saturated model $\mathcal{A}$, define
\[
U(\bar{a}/\mathcal{A}) = \min\{\alpha : U(\bar{a}/\mathcal{A}) \not\geq \alpha + 1\}
\]
if such an ordinal exists. Then define $U$-rank for arbitrary $\aleph_0$-saturated model $\mathcal{A}$ as
\[
U(\bar{a}/\mathcal{A}) = \min\{U(\bar{a}/\mathcal{A}') : \mathcal{A}' \subset \mathcal{A} \text{ countable } \aleph_0\text{-saturated model}\}.
\]
For finite $\bar{a}$ and a set $A$, define
\[
U(\bar{a}/A) = \sup\{U(\bar{b}/\mathcal{A}) : \text{tp}_w(\bar{b}/A) = \text{tp}_w(\bar{a}/A), \ A \subset \mathcal{A} \text{ and } \mathcal{A} \aleph_0\text{-saturated}\}.
\]
Finally we prove the equation (2.2).

**Proposition 2.44.** Assume that $\mathcal{A}$ is a countable $\aleph_0$-saturated model and both $U(\bar{a}/\mathcal{A})$ and $U(\bar{b}/\mathcal{A})$ are finite. Then we have
\[
U(\bar{a}\bar{b}/\mathcal{A}) \leq U(\bar{a}/\mathcal{A}) + U(\bar{b}/\mathcal{A}).
\]

**Proof.** Let $U(\bar{a}/\mathcal{A}) = n$ and $U(\bar{b}/\mathcal{A}) = m$ for finite $n$ and $m$. We assume the contrary, that $U(\bar{a}\bar{b}/\mathcal{A}) > m + n$. Then by the definition of $U$-rank there are countable and $\aleph_0$-saturated $\mathcal{A}_i$ for $0 \leq i \leq m + n + 1$ such that $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ and $\bar{a}\bar{b} \not\models_{\mathcal{A}_i} \mathcal{A}_{i+1}$ for each $i$. Using Lemma 2.40 we define inductively, starting from $i = m + n + 1$, models $\mathcal{B}_i$ for $m + n + 1 \geq i \geq 0$ as follows:
Let $0 \leq i \leq m + n + 1$. We claim that either $\bar{a} \not \in \mathcal{A}_i \mathcal{B}_{i+1}$ or $\bar{b} \not \in \mathcal{A}_i \mathcal{A}_{i+1}$. To prove the claim, assume that neither holds. By symmetry and monotonicity, $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \bar{a}$. On the other hand by symmetry $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \bar{b}$, and by dominance, $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \mathcal{B}_i$. We get from transitivity that $\mathcal{A}_{i+1} \downarrow_{\mathcal{A}_i} \bar{a} \cup \mathcal{B}_i$, but then $\bar{a} \cup \bar{b} \downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$ by monotonicity and symmetry, a contradiction. This proves the claim.

Now there must be either $n + 1$ indexes $i$ such that $\bar{a} \not \in \mathcal{A}_i \mathcal{B}_{i+1}$ or $m + 1$ indexes $i$ such that $\bar{b} \not \in \mathcal{A}_i \mathcal{A}_i$. Since $\mathcal{A} \subset \mathcal{A}_i$ and $\mathcal{A} \subset \mathcal{B}_i$ for each $i$, both cases give a contradiction with the definition of $U$-rank. □

2.5. Morley sequences. We also recall the definition of a Morley-sequence and the proof of the following Lemma, which is Lemma 5.3 in [6], proved using Fodor’s Lemma. The formulation of the lemma in [6] was slightly weaker, but the same proof clearly gives also the following statement.

Definition 2.45 (Morley sequence). Suppose $\mathcal{A}$ is an $\mathbb{N}_0$-saturated model. We say that $(\bar{a}_i)_{i<\alpha}$ is a Morley sequence over $\mathcal{A}$ if for each $i < j < \alpha$, $\text{tp}^w(\bar{a}_i/\mathcal{A}) = \text{tp}^w(\bar{a}_j/\mathcal{A})$ and for each $i < \alpha$, $\bar{a}_i \downarrow^w \bigcup_{j<i} \bar{a}_j$.

Lemma 2.46. Let $(\mathbb{K}, \ll_{\mathbb{K}})$ be an $\mathbb{N}_0$-stable finitary AEC with the extension property. Let $E$ be a set, $I$ a sequence of tuples $\bar{a}_i$ such that $i \neq j$ implies $\bar{a}_i \neq \bar{a}_j$, and $\mathbb{N}_0 + |E| \leq \lambda < |I|$. Then there is an $\mathbb{N}_0$-saturated model $\mathcal{A}$ of size $\lambda$ containing $E$ and subsequence $(\bar{a}_i)_{i<\lambda^+} \subset I$ such that it is a Morley sequence over $\mathcal{A}$.

Furthermore, there is finite $E' \subset \mathcal{A}$ such that $\text{tp}^w(\bar{a}_i/\mathcal{A} \cup \bigcup_{j<i} \bar{a}_j)$ does not split over $E'$ for all $i < \lambda^+$. In addition, if $I \cup E \subset \mathcal{B}$ for some $\mathbb{N}_0$-saturated model $\mathcal{B}$, we may choose $\mathcal{A} \subset \mathcal{B}$.

Also the following is proved in Lemma 5.2 of [6].

Lemma 2.47. Let $(\bar{a}_i)_{i<\alpha}$ be a Morley sequence over an $\mathbb{N}_0$-saturated model $\mathcal{A}$. Then for every $n$ and $i_0 < \ldots < i_n < \alpha$ we have that $\text{tp}^w(\bar{a}_{i_0}, \ldots, a_n/\mathcal{A}) = \text{tp}^w(\bar{a}_{i_0}, \ldots, \bar{a}_{i_n}/\mathcal{A})$.

Using Corollary 2.15 instead of the homogeneous model $\mathfrak{M}^*$ we can prove the following two results as Lemma 5.8 and Corollary 5.9 of [6].

Definition 2.48. We say that two sequences $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_j)_{j<\omega}$ are equivalent over $E$, if for every finite $n$ we have that $\text{tp}^g(\bar{a}_0, \ldots, a_n/E) = \text{tp}^g(\bar{b}_0, \ldots, \bar{b}_n/E)$.

Lemma 2.49. Let $(\mathbb{K}, \ll_{\mathbb{K}})$ be an $\mathbb{N}_0$-stable finitary AEC with the extension property. Let $E$ be countable and $(\bar{a}_i)_{i<\omega}$ a sequence of finite tuples. The following are equivalent:

1. $\mathcal{B}_{m+n+1} = \mathcal{A}_{m+n+1}[\bar{b}]$,
2. $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ and $\mathcal{B}_i = \mathcal{A}_i[\bar{b}]$. 

The sequence \((\bar{a}_i)_{i<\omega}\) is \(E\)-equivalent to a sequence \((\bar{b}_i)_{i<\omega}\), which is a Morley sequence over some countable \(\aleph_0\)-saturated model \(\mathcal{A}\) containing \(E\).

The sequence \((\bar{a}_i)_{i<\omega}\) is \(E\)-equivalent to a strongly \(E\)-indiscernible sequence \((\bar{b}_i)_{i<\omega}\).

**Corollary 2.50.** Let \((\mathbb{K}, \preceq)\) be an \(\aleph_0\)-stable finitary AEC with the extension property. Let \(E\) be countable and \(I\) an uncountable sequence of distinct tuples. Then for any \(n < \omega\) there is a subsequence \((\bar{a}_0, \ldots, \bar{a}_{n-1}) \subset I\), which is a beginning of a strongly \(E\)-indiscernible sequence.

**2.6. Bounded closure.** In [6], disjoint amalgamation was used also to prove Lemma 5.20 showing that \(\text{bcl}(\mathcal{A}) = \mathcal{A}\) for a model \(\mathcal{A}\). We get this result for \(\aleph_0\)-saturated models. In [6] we defined a set \(E\) to be bounded, if \(|E| < \lambda\), where \(\lambda\) is the cardinal relater to the monster model \(\mathcal{M}\) in Theorem 2.10. We recall that for a set \(E\)

\[ p_\mathcal{A}(E) = \{ \bar{b} \in \mathcal{M} : \text{tp}^w(\bar{a}/E) = \text{tp}^w(\bar{b}/E) \}, \]

and that

\[ \text{bcl}(E) = \{ \bar{a} \in \mathcal{M} : p_\mathcal{A}(E) \text{ is bounded} \}. \]

**Lemma 2.51.** Let \((\mathbb{K}, \preceq)\) be an \(\aleph_0\)-stable finitary AEC with the extension property. The following are equivalent.

1. \(p_\mathcal{A}(E)\) is bounded
2. \(p_\mathcal{A}(E) \subset \mathcal{A}\) for any \(\aleph_0\)-saturated model \(\mathcal{A}\) containing \(E\).

**Proof.** Item (1) clearly follows from (2). We prove that (2) follows from (1). First we claim that for an \(\aleph_0\)-saturated model \(\mathcal{A}\) and \(\bar{a} \notin \mathcal{A}\), always \(\bar{a} \not\in^* \mathcal{A}\). For any finite \(A_0 \subset A\) there is an automorphism \(f \in \text{Aut}(\mathcal{M}/A_0)\) mapping \(\bar{a}\) into \(\mathcal{A}\). The tuples \(\bar{a}\) and \(f(\bar{a})\) witness that the weak type \(\text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{a})\) splits over \(A_0\). This proves the claim.

We assume to the contrary that \(p_\mathcal{A}(E)\) is bounded but (2) does not hold. Let \(\mathcal{A}\) be an \(\aleph_0\)-saturated model containing \(E\) and \(\bar{b} \in p_\mathcal{A}(E) \setminus \mathcal{A}\). Let \(\lambda = |p_\mathcal{A}(E)|^+\). For any ordinal \(i < \lambda\), we can use the extension property to find \(\bar{b}_i\) such that \(\text{tp}^w(\bar{b}_i/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})\) and \(\bar{b}_i \not\in^* \bigcup_{j<i} \bar{b}_j\). Due to the previous claim \(\bar{b}_i \neq \bar{b}_j\) when \(i \neq j\). But now all \(\bar{b}_i\)'s belong to \(p_\mathcal{A}(E)\), a contradiction.

We defined a type \(\text{tp}^w(\bar{a}/E)\) to be bounded if the set \(p_\mathcal{A}(E)\) is bounded. The previous lemma implies that when \(\mathcal{A}\) is an \(\aleph_0\)-saturated model, a type \(\text{tp}^w(\bar{a}/\mathcal{A})\) is bounded if and only of \(\bar{a} \in \mathcal{A}\).

The reader should now be convinced that we are able to prove all the main results of [6] for the class 2.6.
3. Simplicity

In this section we recall the definitions of strong splitting, independence, simplicity and extensible $U$-rank from [6]. We also recall some results about these and prove some more properties, which were not needed there but will be of use in this paper. For example, we will show that simplicity implies extensible $U$-rank. Throughout this section, if not mentioned otherwise, we will assume that $(\mathcal{K}, \leq_{\mathcal{K}})$ is an $\aleph_0$-stable finitary AEC with the extension property. We will restate these assumptions in each theorem.

In the last subsection 3.2 we will give two equivalent conditions for an $\aleph_0$-stable finitary AEC to have the extension property, and show that the extension property is implied by simplicity and weak categoricity in any uncountable cardinal.

**Definition 3.1 (Lascar strong type).** We say that $\bar{a}$ and $\bar{b}$ have the same Lascar strong type over $E$, written

$$\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E),$$

if $\ell(\bar{a}) = \ell(\bar{b})$ and $E(\bar{a},\bar{b})$ holds for any $E$-invariant equivalence relation $E$ of $\ell(\bar{a})$-tuples with a bounded number of classes.

**Definition 3.2.** We say that $\text{tp}^w(\bar{a}/A)$ Lascar-splits over finite $E$ if there is a strongly $E$-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_0,\bar{a}_1 \in A$ and $\text{tp}^g(\bar{a}_0/E \cup \{\bar{a}\}) \neq \text{tp}^g(\bar{a}_1/E \cup \{\bar{a}\})$.

**Definition 3.3 (Independence).** We say that $\bar{a}$ is independent of $B$ over $C$, write

$$\bar{a} \downarrow C B,$$

if there is finite $E \subset C$ such that for all $D$ containing $C \cup B$ there is $\bar{b}$ such that $\text{tp}^w(\bar{b}/B \cup C) = \text{tp}^w(\bar{a}/B \cup C)$ and $\text{tp}^w(\bar{b}/D)$ does not Lascar-split over $E$. We then write

$$A \downarrow C B,$$

if $\bar{a} \downarrow C B$ for every finite tuple $\bar{a} \in A$.

The definition of simplicity is as in the paper [7]. Simplicity is needed to gain the independence calculus for sets, not only for $\aleph_0$-saturated models. Without simplicity there might not be any notion of independence with the properties of Theorem 3.5, see [8] for an example due to Shelah.

**Definition 3.4 (Simplicity).** We say that $(\mathcal{K}, \leq_{\mathcal{K}})$ is simple if for each $\bar{a}$ and $B$ there is finite $E \subset B$ such that $\bar{a} \downarrow_E B$. 

The main theorem about simplicity, namely Theorem 6.5 of [6], holds also in this context. In [6] it is also specified which (restricted forms) of these properties hold without simplicity. In this section we assume simplicity and do not distinguish which results hold also without simplicity.

**Theorem 3.5.** Let $(\mathbb{K}, \leq_{\mathbb{K}})$ be an $\aleph_0$-stable finitary AEC with the extension property. Assume that $(\mathbb{K}, \leq_{\mathbb{K}})$ is simple. Then, $\downarrow$ satisfies the following properties:

1. **Invariance:** If $A \upharpoonright C B$, then $f(A) \upharpoonright f(C) f(B)$ for an $f \in \text{Aut}(\mathbb{M})$.
2. **Finite character:** $A \upharpoonright C B$ if and only if $\bar{a} \upharpoonright C \bar{b}$ for every finite $\bar{a} \in A$ and $\bar{b} \in B$.
3. **Monotonicity:** If $A \downarrow B D$ and $B \subset C \subset D$ then $A \downarrow B C$ and $A \downarrow C D$.
4. **Local character:** For any finite $\bar{a}$ and any $B$ there exists a finite $E \subset B$ such that $\bar{a} \downarrow E B$.
5. **Extension:** For any $\bar{a}$, $C$ and $B$ containing $C$ there is $\bar{b}$ such that $tp^{u}(\bar{b}/C) = tp^{u}(\bar{a}/C)$ and $\bar{b} \upharpoonright C B$.
6. **Extension for Lascar strong types:** For any finite $C$, $\bar{a}$ and any $B$ containing $C$, there is $\bar{b}$ such that $\text{Lstp}(\bar{b}/C) = \text{Lstp}(\bar{a}/C)$ and $\bar{b} \upharpoonright C B$.
7. **Symmetry:** $A \downarrow C B$ if and only if $B \downarrow C A$.
8. **Transitivity:** Let $B \subset C \subset D$. If $A \downarrow B C$ and $A \downarrow C D$, then $A \downarrow B D$.
9. **Stationarity over $\omega$-saturated models:** Let $\mathcal{A}$ be an $\omega$-saturated model. If $tp^{u}(\bar{a}/\mathcal{A}) = tp^{u}(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow_{\mathcal{A}} B$ and $\bar{b} \downarrow_{\mathcal{A}} B$, then $tp^{u}(\bar{a}/B \cup \mathcal{A}) = tp^{u}(\bar{b}/B \cup \mathcal{A})$.
10. **Stationarity of Lascar strong types:** If $\text{Lstp}(\bar{a}/C) = \text{Lstp}(\bar{b}/C)$, $\bar{a} \downarrow C B$ and $\bar{b} \downarrow C B$, then $tp^{u}(\bar{a}/B \cup C) = tp^{u}(\bar{b}/B \cup C)$.
11. **Pairs lemma:** Let $A \subset B$, $\bar{a} \downarrow_{A} B$ and $\bar{b} \downarrow_{A \cup \bar{a}} B$. Then $\bar{a} \cup \bar{b} \downarrow_{A} B$.

Also as in in [6] we gain the following lemma for an $\aleph_0$-stable finitary AEC with the extension property.

**Lemma 3.6.** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. Then $\bar{a} \downarrow_{\mathcal{A}} B$ if and only if $\bar{a} \downarrow_{\mathcal{A}}^{s} B$.

If $\bar{a} \notin \mathcal{A}$ and $\mathcal{A}$ is an $\aleph_0$-saturated model, we have that $\bar{a} \notin_{\mathcal{A}}^{s} \bar{a}$. From the previous lemma it follows that also $\bar{a} \notin_{\mathcal{A}} \bar{a}$.

We can prove also another property called Left Transitivity, which shows that the result of Pairs Lemma is actually an equivalence.

**Proposition 3.7** (Left transitivity). Let $(\mathbb{K}, \leq_{\mathbb{K}})$ be simple. If $\bar{a} \cup \bar{b} \downarrow_{A} B$, then $\bar{b} \downarrow_{A \cup \bar{a}} B$.

**Proof.** By symmetry, $B \downarrow_{A} \bar{a} \cup \bar{b}$ and then by monotonicity, $B \downarrow_{A \cup \bar{a}} \bar{b}$. By symmetry again, $\bar{b} \downarrow_{A \cup \bar{a}} B$. \(\square\)
We recall also the proof of the following from Proposition 5.29 of [6].

**Proposition 3.8.** Let $\mathcal{A}$ be an $\aleph_0$-saturated model. Then $\mathcal{A}$ is also $\alpha$-saturated, i.e. for every finite subset $A \subset \mathcal{A}$ and $\bar{a}$ the Lascar strong type $\text{Lstp}(\bar{a}/A)$ is realized in $\mathcal{A}$.

3.1. Extensible $U$-rank. We recall the definition of extensible $U$-rank.

**Definition 3.9** (Extensible $U$-rank). We say that $(\mathbb{K}, \preceq_{\mathbb{K}})$ has extensible $U$-rank if for each finite $E$, each $\bar{a}$ and each $\aleph_0$-saturated model $\mathcal{A}$ containing $E$ there is $\bar{b}$ such that $\text{tp}^u(\bar{a}/E) = \text{tp}^u(\bar{b}/E)$ and $U(\bar{b}/\mathcal{A}) = U(\bar{a}/E)$.

In Theorem 6.17 of [6] we obtained the following result.

**Theorem 3.10.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary $\mathcal{AEC}$, stable in $\aleph_0$, with extension property and extensible $U$-rank. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple.

Here we add that simplicity also implies extensible $U$-rank. The proof is again similar to the one in [7]. First we need a technical lemma for the proof.

**Lemma 3.11.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let $\mathcal{A}$ be countable $\aleph_0$-saturated model and $E \subset \mathcal{A}$ finite such that $\mathcal{A} \downarrow_E \bar{a}$. Let $B$ be any set. Then there is $f \in \text{Aut}(\mathcal{M}/E \cup \bar{a})$ such that $f(\mathcal{A}) \downarrow_E \bar{a} \cup B$.

**Proof.** Write $\mathcal{A} = \bigcup_{n<\omega} A_n$ as an increasing union of finite sets such that $E \subset A_0$. Then $A_n \downarrow_E \bar{a}$ for each $n < \omega$. We define by induction on $n$ mappings $f_n \in \text{Aut}(\mathcal{M}/E \cup \bar{a})$ such that $m < n$ implies $f_n \upharpoonright A_m = f_m \upharpoonright A_m$ and $f_n(A_n) \downarrow_E \bar{a} \cup B$ for each $n < \omega$.

First we get $f_0$ from extension. Assume we have defined $f_n$. By invariance $f_n(A_n+1) \downarrow_E \bar{a}$ and by Left Transitivity we get that $f_n(A_{n+1}) \downarrow_{E \cup f_n(A_n)} \bar{a}$. Again by extension there is $g \in \text{Aut}(\mathcal{M}/E \cup f_n(A_n) \cup \bar{a})$ such that $g(f(A_{n+1})) \downarrow_{E \cup f(A_n)} B \cup \bar{a}$. By induction and Pairs Lemma also $g(f_n(A_{n+1})) \downarrow_E B \cup \bar{a}$. We can take $f_{n+1} = g \circ f_n$.

Finally $\bigcup_{n<\omega} f_n \upharpoonright A_n : \mathcal{A} \to \mathcal{M}$ extends to an automorphism $f' \in \text{Aut}(\mathcal{M}/E)$. By the construction $f'(\mathcal{A}) \downarrow_E B \cup \bar{a}$. But now $f' \circ f^{-1}_n \in \text{Aut}(\mathcal{M}/f'(A_n))$ maps $\bar{a}$ to $f'(\bar{a})$ for each $n < \omega$, and we get that $\text{tp}^u(\bar{a}/f'(\mathcal{A})) = \text{tp}^u(f'(\bar{a})/f'(\mathcal{A}))$. By Theorem 2.20 there is $f'' \in \text{Aut}(\mathcal{M}/f'(\mathcal{A}))$ such that $f''(f(\bar{a})) = \bar{a}$. Since $f''(f'(\mathcal{A})) = f'(\mathcal{A})$, we can take $f = f'' \circ f' \in \text{Aut}(\mathcal{M}/E \cup \bar{a})$. \hfill $\square$

We also recall the proofs of the following facts from [6].

**Theorem 3.12.** Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an $\aleph_0$-stable finitary $\mathcal{AEC}$ with the extension property. Assume that $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ are $\aleph_0$-saturated. Then
Lemma 3.13. Assume that $E$ is finite. Then

$$U(\bar{a}/E) = \sup\{U(\bar{a}/\mathcal{A}) : \mathcal{A} \aleph_0\text{-saturated and countable, } E \subseteq \mathcal{A}\}.$$ 

The following proposition is similar to the one in [7].

Proposition 3.14. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let $C$ be finite, $\mathcal{A}$ an $\aleph_0$-saturated model containing $E$ and $\bar{b}$ a tuple. Then $\bar{b} \downarrow_E \mathcal{A}$ implies that $U(\bar{b}/E) = U(\bar{b}/\mathcal{A})$.

Proof. By definition we have that

$$U(\bar{b}/\mathcal{A}) = \min\{U(\bar{b}/\mathcal{A}'): \mathcal{A}' \subseteq \mathcal{A}, \mathcal{A}' \aleph_0\text{-saturated and countable}\}.$$ 

Let $\mathcal{A}'$ be countable and $\aleph_0$-saturated such that $U(\bar{b}/\mathcal{A}') = U(\bar{b}/\mathcal{A})$. By Theorem 3.12(1) we may assume that $E \subseteq \mathcal{A}'$. We need to show that $U(\bar{b}/E) = U(\bar{b}/\mathcal{A}')$.

We prove that for every countable $\aleph_0$-saturated model $\mathcal{A}$ containing $E$ such that $\bar{b} \downarrow_E \mathcal{A}$, $U(\bar{b}/E) \geq \alpha$ implies that $U(\bar{b}/\mathcal{A}) \geq \alpha$. It is enough to prove this for successor ordinals. Assume that $U(\bar{b}/E) \geq \alpha + 1$. By Lemma 3.13 there is some countable $\aleph_0$-saturated $\mathcal{B}$ such that $E \subseteq \mathcal{B}$ and $U(\bar{b}/E) = U(\bar{b}/\mathcal{B})$. By symmetry, $\mathcal{A} \downarrow_E \bar{b}$ and we get from Lemma 3.11 an automorphism $f \in \text{Aut}(\mathcal{M}/E \cup \bar{b})$ such that $f(\mathcal{A}) \downarrow_E \mathcal{B} \cup \bar{b}$. Since $U(\bar{b}/\mathcal{A}) = U(\bar{b}/f(\mathcal{A}))$, we may assume that $\mathcal{A} \downarrow_E \mathcal{B} \cup \bar{b}$. From monotonicity and symmetry we get that $\bar{b} \downarrow_{\mathcal{B}} \mathcal{A} \cup \mathcal{B}$.

Now let $\mathcal{B}'$ be countable and $\aleph_0$-saturated containing both $\mathcal{A}$ and $\mathcal{B}$. By extension there is $\bar{b}'$ realizing $tp^w(\bar{b}/\mathcal{A} \cup \mathcal{B})$ such that $\bar{b}' \downarrow_{\mathcal{B}'} \mathcal{B}'$. By Theorem 3.12 and since weak type preserves $U$-rank we get that $U(\bar{b}/\mathcal{A}) = U(\bar{b}/\mathcal{A}') \geq U(\bar{b}/\mathcal{B}') = U(\bar{b}/\mathcal{B}) = U(\bar{b}/\mathcal{B}) \geq \alpha + 1$. \qed

Finally we get the result.

Corollary 3.15. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Then $(\mathbb{K}, \preceq_{\mathbb{K}})$ has extensible $U$-rank.

Proof. Let $E$ be finite, $\bar{a}$ a tuple and $\mathcal{A}$ an $\aleph_0$-saturated model containing $E$. By extension, there is $\bar{b}$ such that $tp^w(\bar{b}/E) = tp^w(\bar{a}/E)$ and $\bar{b} \downarrow_E \mathcal{A}$. The previous proposition implies that this $\bar{b}$ is the one needed for the definition of extensible $U$-rank, since now $U(\bar{a}/E) = U(\bar{b}/E) = U(\bar{b}/\mathcal{A})$. \qed

Simplicity implies the result of Theorem 3.12 holds also for finite sets $A \subseteq B$. For this we need another technical lemma.
Lemma 3.16. Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is simple. Assume that \(\bar{a}\) is a tuple, \(A\) a finite set and \(\mathcal{B}\) an \(\aleph_0\)-saturated model containing \(A\). Then there is a countable \(\aleph_0\)-saturated model \(\mathcal{A} \preceq_\mathbb{K} \mathcal{B}\) such that \(A \subseteq \mathcal{A}\) and \(\mathcal{A} \downarrow A \bar{a}\).

Proof. We define finite sets \(B_i \subseteq \mathcal{B}\) such that \(B_0 = A\) and \(B_i \downarrow A \bar{a}\) for each \(i < \omega\) and \(\bigcup_{i<\omega} B_i\) has the \(\aleph_0\)-saturation property of Proposition 2.38. Then we have that \(\mathcal{A} = \bigcup_{i<\omega} B_i \preceq_\mathbb{K} \mathcal{B}\) and \(\mathcal{A} \downarrow A \bar{a}\). Simultaneously we define weak types \(p_i\) with domain \(B_i\) for each \(i < \omega\).

First we let \(B_0 = A \subseteq \mathcal{B}\). Then \(B_0 \downarrow A \bar{a}\) by simplicity and symmetry. Assume we have defined \(B_n\). By, \(\aleph_0\)-stability, there are countably many different types over \(B_n\). Let \((\bar{b}_i^n)_{i<\omega}\) be representatives of all types over \(B_n\). Let \(p_n\) be the type of the catenation of tuples \(\bar{b}_i^n\) for \(i, j \leq n\) over \(B_n\).

Let \(B \subseteq \mathcal{B}\) be finite such that \(\bar{a} \cup B_n \downarrow B \mathcal{B}\). By simplicity and extension there is \(B_{n+1}\) satisfying the type \(p_n\) such that \(B_{n+1} \downarrow B_n \cup B\). Since \(\mathcal{B}\) is \(\aleph_0\)-saturated, we may assume that \(B_{n+1} \subseteq \mathcal{B}\). Now, by symmetry, \(B_{n+1} \downarrow B \bar{a} \cup B\).

By monotonicity, \(B_{n+1} \downarrow B \cup B\bar{a}\) and by transitivity, \(B_{n+1} \downarrow B_n \cup \bar{a}\). We have that \(B_{n+1} \downarrow B_n \cup \bar{a}\) and \(B_n \downarrow A \bar{a}\), and since by Pairs Lemma \(B_{n+1} \downarrow A \bar{a}\).

We can easily see that \(\bigcup_{i<\omega} B_i\) has the \(\aleph_0\)-saturation property, and thus is a model by Proposition 2.38. \(\square\)

Proposition 3.17. Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is simple. Let \(A \subseteq B\) be finite sets. Then

1. \(U(\bar{a}/B) \leq U(\bar{a}/A)\) and
2. \(U(\bar{a}/A) = U(\bar{a}/B)\) if and only if \(\bar{a} \downarrow A B\).

Proof. Item (1) is clear from the definition. We prove item 2. Let \(\mathcal{B}'\) be countable and \(\aleph_0\)-saturated containing \(B\). By extension there is an automorphism \(f \in \text{Aut}(\mathcal{M}/B)\) such that \(f(\bar{a}) \downarrow B \mathcal{B}'\). Denote \(\mathcal{B} = f^{-1}(\mathcal{B}')\). Now \(\mathcal{B}\) is a countable \(\aleph_0\)-saturated model such that \(A \subseteq B \subseteq \mathcal{B}\) and by invariance \(\bar{a} \downarrow B \mathcal{B}\).

Proposition 3.14 says that \(U(\bar{a}/B) = U(\bar{a}/\mathcal{B})\). Furthermore, Lemma 3.16 gives a countable \(\aleph_0\)-saturated \(\mathcal{A} \preceq_\mathbb{K} \mathcal{B}\) such that \(A \subseteq \mathcal{A}\) and \(\mathcal{A} \downarrow A \bar{a}\). By symmetry, \(\bar{a} \downarrow A \mathcal{A}\) and again by Proposition 3.14, \(U(\bar{a}/A) = U(\bar{a}/\mathcal{A})\).

Assume that \(U(\bar{a}/A) = U(\bar{a}/B)\) and thus \(U(\bar{a}/\mathcal{A}) = U(\bar{a}/\mathcal{B})\). Theorem 3.12 gives that \(\bar{a} \downarrow \mathcal{A} \mathcal{B}\), and by transitivity \(\bar{a} \downarrow A \mathcal{B}\) and by monotonicity \(\bar{a} \downarrow A B\).

Then assume that \(\bar{a} \not\in A B\). By the previous reasoning we must have that \(\bar{a} \not\in \mathcal{A} \mathcal{B}\). Then similarly by Theorem 3.12, \(U(\bar{a}/A) > U(\bar{a}/B)\). \(\square\)

The following corollary will be used to prove the existence of \(\bar{f}\)-primary models in section 4.
Corollary 3.18. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple. Let $\bar{a}_i$ be tuples and $A_i$ finite sets such that $A_i \subset A_{i+1}$ and $\text{tp}^w(\bar{a}_{i+1}/A_i) = \text{tp}^w(\bar{a}_i/A_i)$ for each $i < \omega$. Then there is $i < \omega$ such that $\bar{a}_{i+1} \downarrow_{A_i} A_{i+1}$.

Proof. Assume to the contrary that $\bar{a}_{i+1} \not\downarrow_{A_i} A_{i+1}$ for each $i < \omega$. Then by the previous Proposition $U(\bar{a}_i/A_i)_{i<\omega}$ is a strictly decreasing sequence of ordinals, a contradiction. □

3.2. Extension property. We will show that simplicity together with weak $\kappa$-categoricity imply the extension property. First we write two equivalent conditions for an $\aleph_0$-stable finitary AEC to satisfy the extension property. We define a concept of an abstract type $p$. We say that $p$ is a finitely realized weak type over $B$, if $p$ is a collection

$$p = \{\bar{a}_A : A \subset B \text{ finite}\},$$

such that for any finite $A \subset A' \subset B$, $\text{tp}^w(\bar{a}_{A'}/A) = \text{tp}^w(\bar{a}_A/A)$. Two types $p$ and $p'$ are equal if whenever $\bar{a}_A \in p$ and $\bar{a}'_A \in p'$, $\text{tp}^w(\bar{a}_A/A) = \text{tp}^w(\bar{a}'_A/A)$. Also when $B' \subset B$, the restriction $p \upharpoonright B'$ is defined as

$$p \upharpoonright B' = \{\bar{a}_A \in p : A \subset B'\}.$$  

We also say that $p$ is realized if there is $\bar{a}$ such that $\text{tp}^w(\bar{a}/A) = \text{tp}^w(\bar{a}_A/A)$ for each finite $A \subset B$ and $\bar{a}_A \in p$. We recall the following result, which is Lemma 3.13 of [6].

Lemma 3.19. Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an $\aleph_0$-stable finitary AEC. Assume that $\mathcal{A}$ is a model and $|\mathcal{A}| \leq \aleph_1$. Let $p$ be a finitely realized type over $\mathcal{A}$. Then $p$ is realized.

The next lemma shows that for a finitely realized weak type $p$ over a model $\mathcal{A}$ there is a finite $E \subset \mathcal{A}$ such that $p$ does not split over $E$. We define that a finitely realized weak type $p$ over a model $\mathcal{A}$ does not split over $E$, if for every finite $B \subset \mathcal{A}$ and $\bar{a}$ realizing $p \upharpoonright (E \cup B)$, the type $\text{tp}^w(\bar{a}/E \cup B)$ does not split over $E$. By finite character of splitting, this is a reasonable definition.

Lemma 3.20. Let $(\mathbb{K}, \preceq_{\mathbb{K}})$ be an $\aleph_0$-stable finitary AEC. Let $p$ be a finitely realized weak type over a model $\mathcal{A}$. Then there is finite $E \subset \mathcal{A}$ such that $p$ does not split over $E$.

Proof. We assume the contrary. By finite character of splitting we can construct a sequence of countable models $\mathcal{A}_n \preceq_{\mathbb{K}} \mathcal{A}$, $n < \omega$ such that the type $p \upharpoonright \mathcal{A}_{n+1}$ splits over each finite $E \subset \mathcal{A}_n$. Denote $\mathcal{A}' = \bigcup_{n<\omega} \mathcal{A}_n \preceq_{\mathbb{K}} \mathcal{A}$. But now, the type $p \upharpoonright \mathcal{A}'$ is realized and splits over any finite $E \subset \mathcal{A}'$, a contradiction with local character. □
Finally we represent the three equivalent conditions.

**Proposition 3.21.** Assume that \((K, \preceq_K)\) is an \(\aleph_0\)-stable finitary AEC. The following are equivalent:

1. \((K, \preceq_K)\) has the extension property.
2. There exists an infinite cardinal \(\kappa\) such that every finitely realized weak type \(p\) over a weakly \(\kappa\)-saturated model \(\mathcal{A}\) is realized.
3. For any \(\aleph_0\)-saturated model \(\mathcal{A}\) and any \(\bar{a}\) there is finite \(E \subseteq \mathcal{A}\) such that the following holds: For any \(B \supseteq \mathcal{A}\) there is \(\bar{b}\) realizing \(tp^w(\bar{a}/\mathcal{A})\) such that \(tp^w(\bar{b}/B)\) does not split over \(E\).

**Proof.** Clearly (1) implies (3) by local character of splitting. First we show that (2) implies (1). Assume that \(\mathcal{A}\) is \(\aleph_0\)-saturated and \(tp(\bar{a}/\mathcal{A})\) does not split over finite \(E \subseteq \mathcal{A}\). We would like to find a free extension of \(tp^w(\bar{a}/\mathcal{A})\) to a set \(B\) containing \(\mathcal{A}\). By increasing the size of \(B\) if necessary, we may assume that \(B = \mathcal{B}\) is a weakly \(\kappa\)-saturated model for the \(\kappa\) from condition (2). Let \(\mathcal{A}' \preceq_K \mathcal{A}\) be a countable \(\aleph_0\)-saturated model containing \(\mathcal{E}\). By countable extension, for any finite \(A \subseteq \mathcal{B}\) there is \(\bar{a}_A\) realizing \(tp^w(\bar{a}/\mathcal{A}')\) such that \(tp^w(\bar{a}_A/\mathcal{A}' \cup A)\) does not split over \(E\). By stationarity, when \(A \subset A'\) are finite subsets of \(\mathcal{B}\), \(tp^w(\bar{a}_A/\mathcal{A}' \cup A) = tp^w(\bar{a}_A/\mathcal{A}' \cup A)\). Then by (2), there is \(\bar{b}\) realizing the type \(tp^w(\bar{a}_A/A)\) for each finite \(A \subseteq \mathcal{B}\). By finite character of splitting, the type \(tp^w(\bar{b}/\mathcal{A}')\) does not split over \(E\), and again by stationarity, \(tp^w(\bar{b}/\mathcal{A}') = tp^w(\bar{a}/\mathcal{A}')\).

Finally we show that (3) implies (2), where we take \(\kappa = \aleph_0\). Let \(p\) be a finitely realized type over an \(\aleph_0\)-saturated model \(\mathcal{A}\). By Lemma 3.20 there is finite \(A \subset \mathcal{A}\) such that \(p\) does not split over \(A\). Let \(\mathcal{A}' \preceq_K \mathcal{A}\) be countable \(\aleph_0\)-saturated model containing \(A\) and let \(\bar{a}\) realize \(p \upharpoonright \mathcal{A}'\). By (3), there is finite \(E \subset \mathcal{A}'\) and \(\bar{b}\) such that \(\bar{b}\) realizes \(tp^w(\bar{a}/\mathcal{A}')\) and \(tp^w(\bar{b}/\mathcal{A}')\) does not split over \(E\). But now by stationarity, \(\bar{b}\) realizes \(p\).

We say that \((K, \preceq_K)\) is **strongly \(\aleph_0\)-stable** if there are only countably many different Lascar strong types over a countable set \(E\). Strong \(\aleph_0\)-stability implies \(\aleph_0\)-stability, since having the same weak type over a countable set \(E\) is an \(E\)-invariant equivalence relation with a bounded number of classes. In [6], we showed that an \(\aleph_0\)-stable finitary AEC with the extension property is strongly \(\aleph_0\)-stable. We show that also without the extension property, strong \(\aleph_0\)-stability is implied by weak categoricity in some uncountable \(\kappa\). We recall from [6], that whenever \((\bar{a}_i)_{i < \alpha}\) is a strongly \(E\)-indiscernible sequence, we have that \(\text{Lstp}(\bar{a}_i/E) = \text{Lstp}(\bar{a}_j/E)\) for each \(i < j < \alpha\). This is based only on the definitions of strong indiscernibility and the Lascar strong type.
Proposition 3.22. Assume that \((\mathbb{K}, \approx_{\mathbb{K}})\) is a finitary AEC and weakly categorical in an uncountable cardinal \(\kappa\). Then \((\mathbb{K}, \approx_{\mathbb{K}})\) is strongly \(\aleph_0\)-stable.

Proof. Let \(E\) be a countable set and let \((b_i)_{i<\omega_1}\) be a sequence of distinct tuples. We want to find \(i < j < \omega_1\) such that \(\text{Lstp}(b_i/E) = \text{Lstp}(b_j/E)\). Let \(EM(\kappa \cup E)\) be a model as in Proposition 2.13, where \(\kappa\) is interpreted as the linear order \(\kappa\). By weak \(\kappa\)-categoricity, the model \(EM(\kappa \cup E)\) is weakly \(\aleph_1\)-saturated, and then also \(\aleph_1\)-saturated respect to Galois types. We are able to construct an automorphism mapping the sequence \((b_i)_{i<\omega_1}\) into \(EM(\kappa \cup E)\) and fixing \(E\) pointwise. Since having the same Lascar strong type over \(E\) is invariant under automorphisms fixing \(E\), we may assume that \((\bar{b})_{i<\omega_1} \subset SH(\kappa \cup E)\).

Since \(\tau^*\) and \(E\) are countable, we may assume that each \(\bar{b}_j\) is generated similarly and with the same sequence of terms of \(\tau^*\) from a fixed finite subset of \(E\) and some \((\alpha_0, ..., \alpha_n)\) for \(\alpha_0 < ... < \alpha_n < \kappa\). By Lemma 2.18 there is a subsequence \((\bar{b}_{j_k})_{k<\omega_1}\) such that it is a strongly \(E\)-indiscernible sequence. Thus \(\text{Lstp}(\bar{b}_{i_k}/E) = \text{Lstp}(\bar{b}_{j_k}/E)\) for all \(k < p < \omega_1\).

We recall the group \(\text{Saut}(\mathcal{M}/E)\), which is a normal subgroup of \(\text{Aut}(\mathcal{M}/E)\) such that \(f \in \text{Saut}(\mathcal{M}/E)\) if for all \(\bar{a}\), \(\text{Lstp}(\bar{a}/E) = \text{Lstp}(f(\bar{a})/E)\). We say that an \(f \in \text{Saut}(\mathcal{M}/E)\) is a strong automorphism. We also recall that a model \(\mathcal{A}\) is said to be \(\alpha\)-saturated, if for any finite \(E \subset \mathcal{A}\) and \(\bar{a}\), there is \(\bar{b} \in \mathcal{A}\) such that \(\text{Lstp}(\bar{b}/E) = \text{Lstp}(\bar{a}/E)\). Imitating the proofs of similar results in [6], we are able to prove the following.

Lemma 3.23. Assume that \((\mathbb{K}, \approx_{\mathbb{K}})\) is a strongly \(\aleph_0\)-stable finitary AEC.

(1) Let \(E\) be countable. Then \(\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)\) if and only if there is \(f \in \text{Saut}(\mathcal{M}/E)\) such that \(f(\bar{a}) = \bar{b}\).

(2) Each \(\aleph_0\)-saturated model is also \(\alpha\)-saturated.

(3) For countable \(E\) and \(\bar{a} \neq \bar{b}\), \(\text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E)\) if and only if there exists \(n < \omega\), \(\bar{a}_i\) and strongly \(E\)-indiscernible sequences \(I_i\) for \(i \leq n\) such that \(\bar{a}_0 = \bar{a}, \bar{a}_n = \bar{b}\) and \(\bar{a}_i, \bar{a}_{i+1} \in I_i\) for \(i < n\).

(4) Let \(\mathcal{A}\) be an \(\aleph_0\)-saturated model and \(E \subset \mathcal{A}\) finite. Then \(\text{tp}^w(\bar{a}/\mathcal{A})\) Lascar-splits over \(E\) if and only if there are \(\bar{c}, \bar{d} \in \mathcal{A}\) such that \(\text{Lstp}(\bar{c}/E) = \text{Lstp}(\bar{d}/E)\) and \(\text{tp}^w(\bar{c}/E \cup \bar{a}) \neq \text{tp}^w(\bar{d}/E \cup \bar{a})\).

(5) Assume that \(\mathcal{A} \subset \mathcal{B}\) are \(\aleph_0\)-saturated. Let \(\bar{a}\) and \(\bar{b}\) be such that \(\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})\), the type \(\text{tp}^w(\bar{a}/\mathcal{A})\) does not Lascar-split over finite \(E \subset \mathcal{A}\) and the type \(\text{tp}^w(\bar{b}/\mathcal{B})\) does not split over finite \(E' \subset \mathcal{A}\). Then we have \(\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{B})\).

\(^3\text{In the proof of item (3) we should use Corollary 2.15 instead of Corollary 5.9 of [6].}\)
Finally we are able to show the following.

**Proposition 3.24.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a simple finitary AEC and weakly categorical in some uncountable \(\kappa\). Then \((\mathbb{K}, \preceq_{\mathbb{K}})\) has the extension property.

**Proof.** By Proposition 3.22, the class \((\mathbb{K}, \preceq_{\mathbb{K}})\) is strongly \(\aleph_0\)-stable and hence also \(\aleph_0\)-stable. We will show that condition (2) of Proposition 3.21 holds. For this, let \(p\) be a finitely realized type over an \(\aleph_0\)-saturated model \(\mathcal{A}\).

By Lemma 3.20 there is finite \(E \subset \mathcal{A}\) such that \(p\) does not split over \(E\). Then \(p\) does not Lascar-split over \(E\) either. Let \(\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{A}\) be countable and \(\aleph_0\)-saturated such that \(E \subset \mathcal{A}_0\). Then by Lemma 3.19 there is \(\bar{a}\) realizing \(p\restriction_{\mathcal{A}_0}\). But now by simplicity, \(\bar{a} \downarrow_{\mathcal{A}_0} \mathcal{A}_0\) and thus there is finite \(E' \subset \mathcal{A}_0\) and \(\bar{b}\) such that \(\text{tp}^w(\bar{b}/\mathcal{A}_0) = \text{tp}(\bar{a}/\mathcal{A}_0)\) and \(\text{tp}^w(\bar{b}/\mathcal{A})\) does not Lascar-split over \(E'\). But now by (5) of Lemma 3.23, \(\bar{b}\) realizes \(p\restriction \mathcal{B}\) for each countable \(\aleph_0\)-saturated \(\mathcal{A}_0 \preceq_{\mathbb{K}} \mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}\), and thus realizes \(p\). \(\square\)

4. **f-primary models**

In this section we introduce f-primary models. The notion imitates the notion of \(F^f\)-primary in [12]. The existence of f-primary models is implied by simplicity, and they can be used to show that weak categoricity implies every \(\aleph_0\)-saturated model being weakly saturated. Again we will assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is an \(\aleph_0\)-stable finitary AEC with the extension property.

**Definition 4.1** (f-isolated). Let \(\bar{a}\) be a tuple and \(A\) a set. A weak type \(\text{tp}^w(\bar{a}/A)\) is f-isolated over \(E \subset A\) if whenever \(\bar{b}\) realizes \(\text{tp}^w(\bar{a}/E)\), then \(\bar{b} \downarrow_E A\).

We remark that if \(\text{tp}^w(\bar{a}/A)\) is f-isolated over \(E \subset A\) and \(E \subset E' \subset A\), then \(\text{tp}^w(\bar{a}/A)\) is also f-isolated over \(E'\). We recall the \(\aleph_0\)-saturation property: a set \(A\) has the property if for any finite \(B \subset A\) and a tuple \(\bar{b}\) there is \(\bar{a}\) in \(A\) realizing \(\text{tp}^\varnothing(\bar{b}/B)\). By Proposition 2.38, if a set \(A \subset \mathfrak{B}\) has the \(\aleph_0\)-saturation property, then \(\mathcal{A} \in \mathbb{K}\) and \(A \preceq_{\mathbb{K}} \mathfrak{M}\). This is a consequence of finite character and \(\aleph_0\)-stability.

**Definition 4.2** (f-primary). We say that \(\mathcal{A}\) is f-primary over a set \(A\) if for some ordinal \(\xi\) there are tuples \(\bar{a}_i\) and finite sets \(A_i\) for \(i < \xi\) such that

1. the weak type \(\text{tp}^w(\bar{a}_i/A \cup \bigcup_{j<i} \bar{a}_j)\) is f-isolated over \(A_i \subset A \cup \bigcup_{j<i} \bar{a}_j\) and
2. \(\mathcal{A} = A \cup \bigcup_{i<\xi} \bar{a}_i\) has the \(\aleph_0\)-saturation property.

**Lemma 4.3.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is simple. For every tuple \(\bar{a}\), set \(A\) and finite \(B \subset A\) there is \(\bar{b}\) and finite \(A_0 \subset A\) such that \(\text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B)\) and \(\text{tp}^w(\bar{b}/A)\) is f-isolated over \(A_0\).
Proof. Assume that \( \vec{a}, A \) and finite \( B \subset A \) would witness the contrary. We define tuples \( \vec{a}_i \) and finite sets \( A_i \) for \( i < \omega \) to contradict Corollary 3.18. First let \( \vec{a}_0 = \vec{a} \) and \( A_0 = B \). Then assume we have defined \( \vec{a}_i \) and \( A_i \) for \( i \leq n \) such that

1. \( \text{tp}^w(\vec{a}_i/B) = \text{tp}^w(\vec{a}/B) \),
2. sets \( A_i \) are finite and \( A_i \subset A_{i+1} \subset A \),
3. \( \text{tp}^w(\vec{a}_{i+1}/A_i) = \text{tp}^w(\vec{a}_i/A_i) \) and
4. \( \vec{a}_{i+1} \not\subseteq A_i \ A_{i+1} \).

Since we have (1), the type \( \text{tp}^w(\vec{a}_n/A) \) can’t be \( f \)-isolated over finite \( A_n \subset A \). Thus there is a tuple \( \vec{a}_{n+1} \) such that \( \text{tp}^w(\vec{a}_{n+1}/A_n) = \text{tp}^w(\vec{a}_n/A_n) \) but \( \vec{a}_{n+1} \not\subseteq A_n \ A_n \). Furthermore, by finite character of independence, there is finite \( A_{n+1} \subset A \) such that \( \vec{a}_{n+1} \not\subseteq A_n \ A_{n+1} \). We may assume that \( A_n \subset A_{n+1} \). This construction contradicts Corollary 3.18. \( \square \)

Lemma 4.4. Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is simple. For every set \( A \) there is an \( \aleph_0 \)-saturated model \( \mathcal{B} \) of size \(|A|\) such that it is \( f \)-primary over \( A \). Furthermore, if \( \mathcal{B}' \) is \( \aleph_0 \)-saturated model containing \( A \), we can choose such \( \mathcal{B} \) that \( \mathcal{B} \preceq_\mathbb{K} \mathcal{B}' \).

Proof. We prove the last claim. Denote \( |A| = \lambda \). By induction on \( n < \omega \) we define sets \( B_n \subset \mathcal{B}' \) of size \( \lambda \), tuples \( \vec{a}_i^n \in \mathcal{B}' \) and finite sets \( A_i^n \subset \mathcal{B}' \) for \( i < \lambda \). First let \( B_0 = A \).

Assume we have defined \( B_n \). Let \( (\text{tp}^w(\vec{b}_i/D_i))_{i < \lambda} \) enumerate all weak types over finite subsets of \( B_n \). Such enumeration exists by \( \aleph_0 \)-stability. Let \( i < \lambda \) and assume we have defined \( \vec{a}_j^n \) for \( j < i \). We use the previous lemma to find a tuple \( \vec{d}_i^n \) realizing \( \text{tp}^w(\vec{b}_i/D_i) \) and a finite subset \( A_i^n \subset B_n \cup \bigcup_{j < i} \vec{a}_j^n \) such that \( \text{tp}^w(\vec{d}_i^n/B_n \cup \bigcup_{j < i} \vec{a}_j^n) \) is \( f \)-isolated over \( A_i^n \). By \( \aleph_0 \)-saturation, there is \( \vec{a}_i^n \in \mathcal{B}' \) realizing \( \text{tp}^w(\vec{d}_i^n/D_i \cup A_i^n) \). Then also \( \text{tp}^w(\vec{a}_i^n/B_n \cup \bigcup_{j < i} \vec{a}_j^n) \) is \( f \)-isolated over \( A_i^n \). Finally let \( B_{n+1} = B_n \cup \bigcup_{i < \lambda} \vec{a}_i^n \).

Each weak type over a finite subset of \( B_n \) is realized in \( B_{n+1} \). Thus \( \mathcal{B} = \bigcup_{n < \omega} B_n = A \cup \bigcup_{(n, i) \in \omega \times \lambda} \vec{a}_i^n \) has the \( \aleph_0 \)-saturation property. We have that \( \mathcal{B} \) is an \( f \)-primary model over \( A \) and is of size \( \lambda \). \( \square \)

Lemma 4.5. Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is simple. Let \( \mathcal{A} \) be \( \aleph_0 \)-saturated, \( A_0, A_1 \subset \mathcal{A} \) finite and

1. \( \text{tp}^w(\vec{a}_0/\mathcal{A} \cup \vec{b}) \) is \( f \)-isolated over \( A_0 \cup \vec{b} \) and
2. \( \text{tp}^w(\vec{a}_1/\mathcal{A} \cup \vec{b} \cup \vec{a}_0) \) \( f \)-isolated over \( A_1 \cup \vec{b} \cup \vec{a}_0 \).

Then there is finite \( A \subset \mathcal{A} \) such that \( \text{tp}^w(\vec{a}_0 \vec{a}_1/\mathcal{A} \cup \vec{b}) \) is \( f \)-isolated over \( A \cup \vec{b} \).

Proof. Let \( A \subset \mathcal{A} \) be finite such that \( A_0 \cup A_1 \subset A \) and \( \vec{b} \downarrow_A \mathcal{A} \). We claim that this is the set \( A \) required for the lemma. To prove the claim, we assume the contrary.
Let \( \bar{c}_0 \) and \( \bar{c}_1 \) be such that \( \text{tp}^w(\bar{c}_0 \bar{c}_1/A \cup \bar{b}) = \text{tp}^w(\bar{a}_0 \bar{a}_1/A \cup \bar{b}) \) but \( \bar{c}_0 \bar{c}_1 \not\models_{A \cup \bar{b}} \mathcal{A} \). By finite character there is finite \( \bar{d} \in \mathcal{A} \) such that \( \bar{c}_0 \bar{c}_1 \not\models_{A \cup \bar{b}} \bar{d} \). From (1) we get that \( \bar{c}_0 \bar{d} \cup \bar{b} \subseteq A \). Since \( \bar{b} \downarrow_A \mathcal{A} \), we get from symmetry and transitivity that \( \mathcal{A} \downarrow_A \bar{b} \cup \bar{c}_0 \). Similarly \( \mathcal{A} \downarrow_A \bar{b} \cup \bar{c}_0 \).

Let \( f \in \text{Aut}(\mathfrak{M}/A \cup \bar{b}) \) be such that \( f(\bar{c}_0) = \bar{a}_0 \). By Proposition 3.8, there is \( \bar{d}^* \in \mathcal{A} \) such that \( \text{Lstp}(\bar{d}^*/A) = \text{Lstp}(f(\bar{d})/A) \). Since \( \bar{d}^* \in \mathcal{A} \), we have \( \bar{d}^* \downarrow_A \bar{b} \cup \bar{a}_0 \). Furthermore \( \bar{d} \downarrow_A \bar{b} \cup \bar{c}_0 \) implies that \( f(\bar{d}) \downarrow_A \bar{b} \cup \bar{a}_0 \). From stationarity for Lascar strong types we get that \( \text{tp}^w(\bar{d}^*/A \cup \bar{b} \cup \bar{a}_0) = \text{tp}^w(\bar{d}^*/A \cup \bar{b} \cup \bar{a}_0) \) and furthermore \( \text{tp}^w(\bar{d}^* \bar{a}_0/A \cup \bar{b}) = \text{tp}^w(\bar{d} \bar{c}_0/A \cup \bar{b}) \). Let \( a'_1 \) be such that \( \text{tp}^w(\bar{d}^* a'_1/A \cup \bar{b}) = \text{tp}^w(\bar{d} \bar{c}_0/A \cup \bar{b}) \).

Since \( \bar{a}'_1 \bar{a}_0 \not\models_{A \cup \bar{b}} \bar{d}^* \) and \( \bar{a}_0 \downarrow_{A \cup \bar{b}} \bar{d}^* \), Pairs Lemma implies that \( \bar{a}'_1 \not\models_{A \cup \bar{b} \cup \bar{a}_0} \bar{d}^* \), and thus by monotonicity \( \bar{a}'_1 \not\models_{A \cup \bar{b} \cup \bar{a}_0} \bar{d}^* \cup A \). Now we have that \( \text{tp}^w(\bar{a}_1/A \cup \bar{b} \cup \bar{a}_0) = \text{tp}^w(\bar{a}_1/A \cup \bar{b} \cup \bar{a}_0) \) but \( \bar{a}'_1 \not\models_{A \cup \bar{b} \cup \bar{a}_0} \mathcal{A} \), a contradiction with (2).

**Lemma 4.6.** Assume that \( \mathcal{A} = A \cup \bigcup_{i < \xi} \bar{a}_i \) is f-primary over \( A \) and \( \bar{d} \in \mathcal{A} \). Then there is finite \( A' \subseteq A \) and \( i_0 < \ldots < i_n < \xi \) such that \( \bar{d} \subseteq A' \cup \bar{a}_{i_0}, \ldots, \bar{a}_n \) and for each \( 0 \leq p \leq n \) there is finite \( A_p \subseteq A' \cup \bar{a}_0, \ldots, \bar{a}_p \) such that \( \text{tp}^w(\bar{a}_p/A \cup \bar{a}_0, \ldots, \bar{a}_{p-1}) \) is f-isolated over \( A_p \).

**Proof.** We choose ordinals \( i_{n,p} \) for finite \( n,p \) as follows.

First let \( B_0 \subseteq A \) and \( a_{i(0,0)} < \ldots < a_{i(0,k_0)} \) be such that \( B_0 \subseteq A \) and the finite tuple \( \bar{d} \) is included in these. For each \( 0 \leq p \leq k_0 \) there is finite \( A_{(0,p)} \subseteq A \cup \bigcup_{j < i_{(0,p)}} \bar{a}_j \) such that \( \text{tp}^w(\bar{a}_{i(0,p)}/A \cup \bigcup_{j < i_{(0,p)}} \bar{a}_j) \) is f-isolated over \( A_{(0,p)} \). We let \( B_1 \subseteq A \) and \( a_{i(1,0)} < \ldots < a_{i(1,k_1)} \) contain all tuples included in the sets \( A_{(0,p)} \). Then the sets \( A_{(1,p)} \subseteq A \cup \bigcup_{j < i_{(1,p)}} \bar{a}_j \) are defined similarly. We continue this and define a tree \( T \) of ordinals such that \( i_{(n+1,p)} < T i_{(n',p')} \) if \( a_{i_{(n+1,p)}} \) is included in \( A_{(n',p')} \). We get tree whose branches go down the ordinals and each level is finite. Since there can’t be an infinite branch, the tree must be finite. Let \( A' \) contain all sets \( B_m \) for those \( m \) appearing in the tree and let \( \bar{a}_0, \ldots, \bar{a}_n \) contain all \( a_{i_{(m,p)}} \) similarly.

**Proposition 4.7.** Assume that \( (\mathbb{K}, \preceq_{\mathbb{K}}) \) is simple. Let \( \mathcal{A} \) be \( \aleph_0 \)-saturated and \( B \) a set. Let \( \mathcal{A}^* = \mathcal{A} \cup B \cup \bigcup_{i < \xi} \bar{a}_i \) be f-primary over \( \mathcal{A} \cup B \) and let \( \bar{d} \) be a tuple in \( \mathcal{A}^* \). Then there are \( \bar{a} = \bar{a}_{i_0}, \ldots, \bar{a}_{i_n} \) for \( i_0 < \ldots < i_n < \xi \), finite \( A \subseteq \mathcal{A} \) and \( \bar{b} \subseteq B \) such that

1. \( \bar{d} \subseteq A \cup \bar{b} \cup \bar{a} \),
2. \( \text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{b}) \) is f-isolated over \( A \cup \bar{b} \) and
3. the tuple \( \bar{b} \) dominates \( \bar{a} \cup \bar{b} \) over \( \mathcal{A} \).
Proof. Let $A' \cup \bar{b} \subset \mathcal{A} \cup B$, $A_p$ for $0 \leq p \leq n$ and $i_0 < ... < i_n < \xi$ be as in the previous Lemma. We show by induction on $m \leq n$ that there is finite $A'_m \subset \mathcal{A}' \cup \bar{b} \cup \{\bar{a}_{i_0}, ..., \bar{a}_{i_{m-1}}\}$ such that

$$\text{tp}^w(\bar{a}_{i_{m-1}}...a_{i_n}/\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, ..., \bar{a}_{i_{m-1}})$$

is f-isolated over $A'_m$.

When $m = 0$, then $n - m = n$ and the claim is clear. Assume the claim holds for $m$. Then

1. $\text{tp}^w(\bar{a}_{i_{m-1}}...a_{i_n}/\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, ..., \bar{a}_{i_{m-1}})$ is f-isolated over $A'_m$ and
2. $\text{tp}^w(\bar{a}_{i_{m-1}}(\mathcal{A}' \cup \bar{b} \cup \bar{a}_{i_0}, ..., \bar{a}_{i_{m-1}}{-1}))$ is f-isolated over $A_{n(m+1)}$.

Item (1) holds by induction and item (2) by the choice of $A_{n(m+1)}$. We find $A_{m+1}' \subset \mathcal{A} \cup \bar{b} \cup \{\bar{a}_{i_0}, ..., \bar{a}_{i_{m+1}}{-1}\}$ by Lemma 4.5. We can take $A = A'_n$. This proves (1) and (2).

To prove (3), assume to the contrary, that $\bar{c} \downarrow \mathcal{A}$ but $\bar{c} \downarrow \mathcal{A}' \cup \bar{b}$ for some tuple $\bar{c}$. Let $A' \subset \mathcal{A}$ be finite such that $A \subset A'$ and $\bar{c} \downarrow \mathcal{A} \cup \bar{b}$. Since $A \subset A'$, the type $\text{tp}^w(\bar{a}/\mathcal{A} \cup \bar{b})$ is f-isolated over $A' \cup \bar{b}$. By finite character, there is finite $\bar{B}'$ such that $A' \subset \bar{B}' \subset \mathcal{A}$ and $\bar{c} \downarrow \mathcal{A}' \cup \bar{a} \cup \bar{c}$. By symmetry and Lemma 3.6, $\bar{b} \downarrow \mathcal{A} \cup \bar{c}$. We may assume that $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{c})$ does not split over $\bar{B}'$.

Since $\mathcal{A}$ is $\aleph_0$-saturated, there is $f \in \text{Aut}(\mathcal{M}/\mathcal{B}')$ such that $f(\bar{c}) \in \mathcal{A}$. Since $\text{tp}^w(\bar{b}/\mathcal{A} \cup \bar{c})$ does not split over $\bar{B}'$, we may assume that $f(\bar{b}) = \bar{b}$. By $f$-isolation we get that $f(\bar{a}) \downarrow \mathcal{A}' \cup \bar{b}$, and furthermore $f(\bar{a}) \downarrow \mathcal{A}' \cup \bar{b}$. On the other hand, by monotonicity and invariance $f(\bar{c}) \downarrow \mathcal{A}' \cup \bar{b}$, and by symmetry, $\bar{b} \cup \bar{B}' \downarrow \mathcal{A} \cup \bar{c}$. Now by the Pairs Lemma, $f(\bar{a}) \cup \bar{b} \cup \bar{B}' \downarrow \mathcal{A} \cup \bar{c}$.

But since $\bar{c} \downarrow \mathcal{A}' \cup \bar{a} \cup \bar{b} \cup \bar{B}'$, by invariance $f(\bar{c}) \downarrow \mathcal{A}' \cup \bar{a} \cup \bar{b} \cup \bar{B}'$, and by symmetry, $f(\bar{a}) \cup \bar{b} \cup \bar{B}' \downarrow \mathcal{A} \cup \bar{c}$, a contradiction. \qed

By finite character we gain the following corollary.

Corollary 4.8. Let $\mathcal{B} = \mathcal{A} \cup B \cup \{\bar{a}_i\}_{i<\xi}$ be an $f$-primary model over $\mathcal{A} \cup B$, where $\mathcal{A}$ is an $\aleph_0$-saturated model. Then $\mathcal{B}$ dominates $\mathcal{B}$ over the model $\mathcal{A}$.

We prove here also an easy remark using Pairs Lemma.

Remark 4.9. Assume that $(\mathbb{K}, \prec_k)$ is simple. Assume that $\bar{a}_p \downarrow \mathcal{A} \cup \mathcal{B} \cup \{\bar{a}_i\}_{i<\xi}$ for all $0 \leq p \leq n$. Then $\bar{a}_0...\bar{a}_n \downarrow \mathcal{A} \cup \mathcal{B}$.

Proof. By induction on $p$ we prove that

$$\bar{a}_{n-p}...\bar{a}_n \downarrow \mathcal{A} \cup \mathcal{B} \cup \{\bar{a}_i\}_{i<\xi \leq \xi - p}. \tag{4.9}$$

First when $p = 0$, $n - p = n$, and the claim follows from the assumption. Then assume that the claim holds for $p$. In addition we have that $\bar{a}_{n-(p+1)} \downarrow \mathcal{A} \cup \mathcal{B} \cup \{\bar{a}_i\}_{i<\xi \leq \xi - (p+1)}$, and thus the claim for $p + 1$ follows from Pairs Lemma. \qed
4.1. **Categoricity transfer.** In the next proposition we prove the "weak categoricity transfer" for $\aleph_0$-saturated models.

**Proposition 4.10.** Assume that $(\mathbb{K}, \leq_k)$ a simple finitary AEC with the extension property, weakly categorical in some uncountable cardinal $\kappa$. Assume that $\mathcal{A}$ is an uncountable $\aleph_0$-saturated model. Let $B \subset \mathcal{A}$ be such that $|B| < |\mathcal{A}|$ and let $\bar{d} \in \mathfrak{M}$. Then the type $\text{tp}^w(\bar{d}/B)$ is realized in $\mathcal{A}$.

**Proof.** It is enough to prove the claim for all $\mathcal{A}$ such that $|\mathcal{A}|$ is a successor cardinal. If $|\mathcal{A}|$ is a limit, there is an $\aleph_0$-saturated model $B \subset \mathcal{A}' \equiv_{\aleph_0} \mathcal{A}$ of size $|B|^+$, and $\text{tp}^w(\bar{d}/B)$ is realized in $\mathcal{A}'$.

By Lemma 2.46 there is a Morley sequence $(b_i)_{i<\aleph_1} \subset \mathcal{A}$ over an $\aleph_0$-saturated model $\mathcal{B} \subset \mathcal{A}$ containing $B$. Also there is finite $E \subset \mathcal{B}$ such that $\text{tp}^w(b_i/\mathcal{B} \cup \bigcup_{j<i} b_j)$ does not split over $E$ for all $i < \aleph_1$. By local character there is finite $E' \subset \mathcal{B}$ such that $\bar{d} \downarrow_{E'} \mathcal{B}$. Let $\mathcal{C} \equiv_{\aleph_0} \mathcal{B}$ be countable and $\aleph_0$-saturated model containing $E \cup E'$. Then $\bar{d} \downarrow_{\mathcal{C}} \mathcal{B}$ and $b_i \downarrow_{\mathcal{C}} \mathcal{B} \cup \bigcup_{j<i} b_j$ for all $i < \aleph_1$. We show that $\text{tp}^w(\bar{d}/\mathcal{B})$ is realized in $\mathcal{A}$. Using extension we continue the Morley sequence to $(b_i)_{i<\kappa}$, where $\kappa$ is the (weak) categoricity cardinal. Let

$$
\mathcal{C}^* = \mathcal{C} \cup \bigcup_{i<\kappa} b_i \cup \bigcup_{j<\xi} \bar{a}_j
$$

be $f$-primary over $\mathcal{C} \cup \bigcup_{i<\kappa} b_i$. By weak categoricity the model $\mathcal{C}^*$ is $\aleph_1$-saturated and thus the weak type $\text{tp}^w(\bar{d}/\mathcal{C}^*)$ is realized in $\mathcal{C}^*$. We find finite $A \subset \mathcal{C}$, $\bar{b} = b_{i_0} \cup \ldots \cup b_{i_m}$, $i_0 < \ldots < i_m < \kappa$ and $\bar{a} = \bar{a}_{j_0}, \ldots, \bar{a}_{j_n}$, $j_0 < \ldots < j_n < \xi$ as in Proposition 4.7. Denote $\bar{b}^* = b_{j_0}, \ldots, b_{j_n} \in \mathcal{A}$. By Lemma 2.47 and since $\mathcal{C}$ is a countable model, there is an automorphism $f \in \text{Aut}(\mathfrak{M}/\mathcal{C})$ mapping $b_{i_k}$ to $b_k$ for each $0 \leq k \leq m$. Proposition 3.8 gives $\bar{a}^* \in \mathcal{A}$ such that

$$
\text{Lstp}(\bar{a}^*/A \cup \bar{b}^*) = \text{Lstp}(f(\bar{a})/A \cup \bar{b}^*).
$$

By (2) of Proposition 4.7 and invariance, $\text{tp}^w(f(\bar{a})/\mathcal{C} \cup \bar{b}^*)$ is $f$-isolated over $A \cup \bar{b}^*$ and thus $\bar{a}^* \downarrow_{A,\bar{b}^*} \mathcal{C}$. By stationarity of Lascar strong types $\bar{a}^*$ realizes $\text{tp}^w(f(\bar{a})/\mathcal{C} \cup \bar{b}^*)$. Then (3) of Proposition 4.7 gives that $\bar{b}^*$ dominates $\bar{a}^* \cup \bar{b}^*$ over $\mathcal{C}$. But we had that $b_i \downarrow_{\mathcal{C}} \mathcal{B} \cup \bigcup_{j<i} b_j$ for all $i < \aleph_1$, and thus by Remark 4.9, $\bar{b}^* \downarrow_{\mathcal{C}} \mathcal{B}$. Using symmetry we gain that $\bar{a}^* \cup \bar{b}^* \downarrow_{\mathcal{C}} \mathcal{B}$.

We had that $\text{tp}^w(\bar{d}/\mathcal{C})$ was realized in $\bar{a} \cup \bar{b} \cup A$, where $A \subset \mathcal{C}$, and $\text{tp}^w(\bar{a} \bar{b}/\mathcal{C}) = \text{tp}^w(f(\bar{a}) \bar{b}^*/\mathcal{C}) = \text{tp}^w(\bar{a} \bar{b}^*/\mathcal{C})$. We claim that $\text{tp}^w(\bar{d}/\mathcal{B})$ is realized in $\bar{a}^* \cup \bar{b}^* \cup A \subset \mathcal{A}$. But this follows from stationarity over $\aleph_0$-saturated models, since $\bar{d} \downarrow_{\mathcal{C}} \mathcal{B}$ and $\bar{a}^* \bar{b}^* \downarrow_{\mathcal{C}} \mathcal{B}$. □
Finally we state the main theorem. When \((\mathbb{K}, \preceq)\) is an AEC, we define 
\[((\mathbb{K})^{\omega}, \preceq)\) = \{\mathcal{A} \in \mathbb{K} : \mathcal{A} \text{ is } \aleph_0\text{-saturated}\}.

If \((\mathbb{K}, \preceq)\) is an \(\aleph_0\)-stable finitary AEC with the extension property, then clearly also the class \(((\mathbb{K})^{\omega}, \preceq)\) is. We formulate the ‘weak categoricity transfer’ as follows.

**Theorem 4.11.** Assume that \((\mathbb{K}, \preceq)\) is a simple finitary AEC and weakly categorical in some uncountable cardinal \(\kappa\). Then

1. \(((\mathbb{K})^{\omega}, \preceq)\) is weakly categorical in each uncountable \(\kappa\) and
2. \((\mathbb{K}, \preceq)\) is weakly categorical in each \(\lambda\) such that \(\lambda \geq \min\{\kappa, H\}\).

**Proof.** By Theorem 2.24 and Proposition 3.24, the class \((\mathbb{K}, \preceq)\) is also \(\aleph_0\)-stable and has the extension property. The first claim follows from Proposition 4.10. For the second claim, we need to show that each model of size \(\geq \min\{\kappa, H\}\) is \(\aleph_0\)-saturated. If \(\kappa \leq H\), this is clear, since the model of size \(\kappa\) is \(\aleph_0\)-saturated. But also each model of size \(\geq H\) is \(\aleph_0\)-saturated by Lemma 2.26. \(\square\)

We recall the definition of tameness. The class \((\mathbb{K}, \preceq)\) is tame (or \(\text{LS}(\mathbb{K})\)-tame), if whenever \(\mathcal{A}\) is a model and 
\[\text{tp}^g(\bar{a}/\mathcal{A}) \neq \text{tp}^g(\bar{b}/\mathcal{A}),\]
there is a submodel \(\mathcal{B} \preceq \mathcal{A}\) of size \(\text{LS}(\mathbb{K})\) such that 
\[\text{tp}^g(\bar{a}/\mathcal{B}) \neq \text{tp}^g(\bar{b}/\mathcal{B}).\]

With tameness the result of Theorem 2.20 generalizes as follows.

**Theorem 4.12.** Assume that \((\mathbb{K}, \preceq)\) is an \(\aleph_0\)-stable tame finitary AEC and \(\mathcal{A}\) is a model. Then \(\text{tp}^w(\bar{a}/\mathcal{A}) = \text{tp}^w(\bar{b}/\mathcal{A})\) if and only if \(\text{tp}^g(\bar{a}/\mathcal{A}) = \text{tp}^g(\bar{b}/\mathcal{A})\).

We recall that an \(\aleph_0\)-stable and tame finitary AEC always has the extension property. The previous theorem guarantees that unions of types over models behave well, and thus the condition (2) of Proposition 3.21 holds. This was stated in [6] as Theorem 4.13.

**Theorem 4.13.** Assume that \((\mathbb{K}, \preceq)\) is an \(\aleph_0\)-stable tame finitary AEC. Then \((\mathbb{K}, \preceq)\) has the extension property.

Let \(\kappa\) be an uncountable cardinal. By Corollary 2.35, \(\kappa\)-categoricity always implies weak \(\kappa\)-categoricity for an \(\aleph_0\)-stable finitary AEC with the extension property. Now Theorem 4.12 together with Lemma 2.22 imply that if \((\mathbb{K}, \preceq)\) is also tame, weak \(\kappa\)-categoricity is equivalent with \(\kappa\)-categoricity. As a corollary we get the following.
**Corollary 4.14.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a simple tame finitary AEC categorical in some uncountable \(\kappa\). Then

1. \(((\mathbb{K}^\omega, \preceq_{\mathbb{K}}))\) is categorical in each uncountable \(\kappa\) and
2. \((\mathbb{K}, \preceq_{\mathbb{K}})\) is categorical in each \(\lambda\) such that \(\lambda \geq \min\{\kappa, H\}\).

We remark that if \((\mathbb{K}, \preceq_{\mathbb{K}})\) is \(\aleph_0\)-stable, simple, tame, finitary AEC, then the class \((\mathbb{K}^\omega, \preceq_{\mathbb{K}})\) has the full categoricity transfer. That is, if \((\mathbb{K}^\omega, \preceq_{\mathbb{K}})\) is categorical in one uncountable \(\kappa\), it is categorical in each uncountable \(\kappa\). We have to assume \(\aleph_0\)-stability to gain that \(\text{LS}(\mathbb{K}^\omega) = \aleph_0\), since categoricity in \((\mathbb{K}^\omega, \preceq_{\mathbb{K}})\) does not necessarily imply \(\aleph_0\)-stability for \((\mathbb{K}, \preceq_{\mathbb{K}})\).

**References**


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PAPER III

SUPERSTABILITY IN SIMPLE FINITARY AEC
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Submitted.
SUPERSTABILITY IN SIMPLE FINITARY AEC

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Abstract. We continue the study of finitary abstract elementary classes beyond \( \aleph_0 \)-stability. We suggest one possible notion of superstability for simple finitary AECs, and derive from this notion several good properties for independence. We also study constructible models and the behaviour of Galois types and weak Lascar strong types in this context.

We show that superstability is implied by a-categoricity in a suitable cardinal. As an application we prove the following theorem: Assume that \((K, \preceq_K)\) is a simple, tame, finitary AEC, a-categorical in some cardinal \(\kappa\) above the Hanf number such that \(\text{cf}(\kappa) > \omega\). Then \((K, \preceq_K)\) is a-categorical in each cardinal above the Hanf number.

1. Introduction

Saharon Shelah developed the context of abstract elementary classes as a platform to study classification theory for non-elementary classes. In this context one does not study structures in any specific language, but a class \(K\) of structures of the same similarity type with an abstract elementary substructure - relation \(\preceq_K\). This framework is very general, and one might need to refine the axioms of the class to generalize machinery from stability theory for AEC. Several different contexts have been studied, and most of them assume at least amalgamation, see [15], [16], [17], [18], [2] or [1]. We introduced the context of finitary abstract elementary classes. We assume amalgamation, joint embedding and arbitrarily large models in order to work inside a monster model. In addition we assume the Löwenheim-Skolem number being countable and a property we call finite character. When \(\mathcal{A}\) and \(\mathcal{B}\) are models in the class \(K\), finite character says that we can detect whether \(\mathcal{A} \preceq_K \mathcal{B}\) by looking at only finite tuples \(\bar{a} \in \mathcal{A}\) and checking whether the Galois type of the tuple \(\bar{a}\) in \(\mathcal{A}\) agrees with its Galois type in \(\mathcal{B}\). The main non-elementary examples of finitary classes are homogeneous classes, see [13] or [10], and excellent classes, see [14] or [11].
In the papers [8] and [7] we studied the $\aleph_0$-stable case. (See also [6].) We introduced a notion of weak type and weak $\lambda$-stability for a cardinal $\lambda$. We also studied a notion of strong type called *Lascar strong type*, written $\text{Lstp}$, which is the equivalence class of a tuple in the finest invariant equivalence relation with a bounded number of equivalence classes. We defined a notion of independence with a built-in extension property in the style of [10]. We also found useful the concept of *simplicity*, which is the property that $\downarrow$ satisfies $\bar{a} \downarrow_A A$ for all tuples $\bar{a}$ and *finite* sets $A$. In the $\aleph_0$-stable case, simplicity guarantees that we have the independence calculus for all sets, not only for $\aleph_0$-saturated models. This approach generalizes the one in [9] for excellent classes.

The main interest in this paper is again a notion of independence. We find the obvious notion of superstability, namely weak stability in large enough cardinals, insufficient to gain good behaviour for the notion of independence. We call this notion weak superstability and take as the main notion the following.

**Definition 1.1 (Superstability).** We say that the class $(\mathbb{K}, \preceq_\mathbb{K})$ is superstable if it is weakly stable in at least one cardinal and the following holds.

Let $(A_n)_{n<\omega}$ be an increasing sequence of finite sets such that $\bigcup_{n<\omega} A_n$ is a model, and let $\bar{a}$ be a tuple. Then there is $n < \omega$ such that $\bar{a} \downarrow_{A_n} A_{n+1}$.

The properties of the notion of independence under superstability are collected in Theorems 3.5 and 3.13. In Theorem 3.5 we study a superstable simple finitary $\mathbb{AEC}$. In Theorem 3.13 we assume also the *Tarski-Vaught property* and gain all the usual properties of non-forking of complete types. The Tarski-Vaught property makes possible to have countable constructible models. It says that we have countably many 'formulas' such that each set which is 'existentially closed' relative to those is a $\mathbb{K}$-elementary substructure of the monster model. We also prove that $\aleph_0$-stable simple finitary classes are superstable (Corollary 3.28) and have the Tarski-Vaught property (Remark 3.9).

The most important notion of type in the context of $\mathbb{AEC}$ is Galois type. The notion was introduced by Shelah, and named as Galois type by Grossberg in [2]. In our context two tuples $\bar{a}$ and $\bar{b}$ have the same Galois type over a set $A$, written $\text{tp}^A(\bar{a}/A) = \text{tp}^A(\bar{b}/A)$, if there is an automorphism of the monster model mapping $\bar{a}$ to $\bar{b}$ and fixing $A$ pointwise. The behaviour of these types is a key question in model theory.

Grossberg and VanDieren have studied abstract elementary classes with amalgamation and $\mu$-tameness for some $\mu$, see [5], [4] and [3]. The class $(\mathbb{K}, \preceq_\mathbb{K})$ is said

\[\text{In } [8] \text{ and } [7] \text{ we actually studied an a priori stronger notion but we will see that the notions agree under } \aleph_0 \text{-stability.}\]
to be $\mu$-tame, if for any tuples $\bar{a}$ and $\bar{b}$ and a model $\mathcal{A}$, $\text{tp}^g(\bar{a}/\mathcal{A}) \neq \text{tp}^g(\bar{b}/\mathcal{A})$ implies that there is a submodel $\mathcal{A}_0 \preceq_{\mathcal{K}} \mathcal{A}$ such that $|\mathcal{A}_0| \leq \mu$ and $\text{tp}^g(\bar{a}/\mathcal{A}_0) \neq \text{tp}^g(\bar{b}/\mathcal{A}_0)$. This assumption implies many good properties for an abstract elementary class, for example we gain upwards categoricity transfer from a successor cardinal $\kappa^+ > \max\{\text{LS}(\mathcal{K})^+, \mu\}$. However, in many examples Galois types have finite character, that is, if the Galois type of $\bar{a}$ and $\bar{b}$ differ over a set $A$, there is some finite subset $A_0 \subset A$ such that their types differ already over $A_0$. Elementary classes as well as homogeneous classes have this property. Also in excellent classes the same holds when $A$ is assumed to be a model, and in $\aleph_0$-stable finitary classes when $A$ is assumed to be a countable model.

We take as our basic notion of type the weak Lascar strong type, which has finite character by definition. Two tuples $\bar{a}$ and $\bar{b}$ have the same weak Lascar strong type over $A$, written $\text{Lstp}^w(\bar{a}/A) = \text{Lstp}^w(\bar{b}/A)$, if for all finite $A_0 \subset A$ we have that $\text{Lstp}(\bar{a}/A_0) = \text{Lstp}(\bar{b}/A_0)$. We study the relation between these types and Galois types in simple finitary classes. With superstability and the Tarski-Vaught property we obtain that when $A$ is a countable set and $\text{tp}^g(\bar{a}/A) \neq \text{tp}^g(\bar{b}/A)$, there is finite $A_0 \subset A$ such that $\text{Lstp}(\bar{a}/A_0) \neq \text{Lstp}(\bar{b}/A_0)$ (Theorem 3.19). If we assume also $\aleph_0$-tameness, the same holds when $A$ is an arbitrary model (Theorem 3.20), and furthermore if $A$ is an $a$-saturated model, we find finite $A_0 \subset A$ such that $\text{tp}^g(\bar{a}/A_0) \neq \text{tp}^g(\bar{b}/A_0)$ (Theorem 3.21). A model $\mathcal{A}$ is $a$-saturated if every Lascar strong type over a finite subset is realized in $\mathcal{A}$.

In the $\aleph_0$-stable case, the class of $\aleph_0$-saturated models of $\mathcal{K}$, written $\mathcal{K}_\omega$, is an interesting subclass of $\mathcal{K}$. Splitting behaves well in this class and we have the full categoricity transfer in $(\mathcal{K}_\omega, \preceq_{\mathcal{K}})$, when $(\mathcal{K}, \preceq_{\mathcal{K}})$ is an $\aleph_0$-stable and $\aleph_0$-tame simple finitary class (Corollary 4.14(1) of [7]). In this paper we study the class $(\mathcal{K}_a, \preceq_{\mathcal{K}})$, where $\mathcal{K}_a$ is the class of $a$-saturated models of $\mathcal{K}$. Note that when $(\mathcal{K}, \preceq_{\mathcal{K}})$ is a finitary class, the class $(\mathcal{K}_a, \preceq_{\mathcal{K}})$ is an abstract elementary class but not necessarily finitary, since its Löwenheim-Skolem number might be uncountable. We define $a$-categoricity meaning categoricity for the class $(\mathcal{K}_a, \preceq_{\mathcal{K}})$, and show that $a$-categoricity in certain cardinals implies superstability for $(\mathcal{K}, \preceq_{\mathcal{K}})$. In section 4 we define an isolation notion for weak Lascar strong type and a concept of an $a$-primary model. We prove an $a$-categoricity transfer result and state some open questions.

We assume the reader to be familiar with the notions of abstract elementary classes and the most common concepts in stability theory for these, like amalgamation, Galois type and the monster model. We also refer to the results in [8] and [7] without proof.
2. Independence

We studied in [7] finitary AEC \((\mathcal{K}, \preceq_{\mathcal{K}})\), which are abstract elementary classes with Löwenheim-Skolem number \(\aleph_0\), amalgamation, joint embedding, arbitrarily large models and finite character. Models in these classes are models of a countable vocabulary \(\tau\). First we defined a notion of a Galois type over the empty set for a tuple \(\bar{a}\) in a model \(\mathcal{A}\), written \(tp^g(\bar{a}/\emptyset, \mathcal{A})\), such that

\[
\text{if there is } C \in \mathcal{K} \text{ and } \mathcal{K}\text{-embeddings } f : \mathcal{A} \to C \text{ and } g : \mathcal{B} \to C \text{ such that } f(\bar{a}) = g(\bar{b}). \]

Then we defined finite character to be the following property.

**Assumption 2.1** (Finite character). Assume that \(A \subset B\) are models and for all tuples \(\bar{a} \in A\),

\[
\text{tp}^g(\bar{a}/\emptyset, \mathcal{A}) = \text{tp}^g(\bar{a}/\emptyset, \mathcal{B}),
\]

then \(\mathcal{A}\) is a \(\mathcal{K}\)-elementary submodel of \(\mathcal{B}\).

A useful consequence of the finite character property is that, when \(\mathcal{A} \preceq_{\mathcal{K}} \mathcal{B}\) and \(f : \mathcal{A} \to \mathcal{B}\) is an embedding, then \(f\) is a \(\mathcal{K}\)-elementary embedding if and only if \(f\) preserves the Galois types of finite tuples, i.e.

\[
\text{tp}^g(\bar{a}/\emptyset, \mathcal{B}) = \text{tp}^g(f(\bar{a})/\emptyset, \mathcal{B})
\]

for all tuples \(\bar{a} \in \mathcal{A}\). With the usual Jónsson-Fraïssé construction we obtain the following theorem.

**Theorem 2.2** (Monster model). Let \(\mu\) be a cardinal. There is \(\mathcal{M} \in \mathcal{K}\) such that:

1. **Universality:** For all \(\mathcal{A} \in \mathcal{K}\) such that \(|\mathcal{A}| < \mu\), there is a \(\mathcal{K}\)-embedding \(f : \mathcal{A} \to \mathcal{M}\).
2. **\(\mathcal{K}\)-homogeneity:** For all \(\mathcal{A} \preceq_{\mathcal{K}} \mathcal{M}\) such that \(|\mathcal{A}| < \mu\) and mappings \(f : \mathcal{A} \to \mathcal{M}\) such that for all finite tuples \(\bar{a} \in \mathcal{A}\)

\[
\text{tp}^g(\bar{a}/\emptyset, \mathcal{M}) = \text{tp}^g(f(\bar{a})/\emptyset, \mathcal{M}),
\]

there is \(g \in \text{Aut}(\mathcal{M})\) extending \(f\).

We say that a set \(A \subset \mathcal{M}\) is \(\mathcal{M}\)-bounded, if \(|A| < \mu\).

We will always assume that all sets we consider are contained in a monster model \(\mathcal{M}\), and are \(\mathcal{M}\)-bounded. We say that \(\mathcal{A}\) is a model, if \(\mathcal{A} \in \mathcal{K}\) and \(\mathcal{A} \preceq_{\mathcal{K}} \mathcal{M}\).

We will only consider such monster models \(\mu\) is a limit cardinal. Finite character was needed to gain the stronger version of item (2), and without finite character it could be defined as follows:
(2') For all $A \preceq_{\mathbb{K}} M$ such that $|A| < \mu$ and $\mathbb{K}$-elementary $f : A \rightarrow M$, there is $g \in \text{Aut}(M)$ extending $f$.

The monster models are $\mu$-saturated in the following sense: if $M \preceq_{\mathbb{K}} M'$ are two monster models and $A \subset M$, $B \subset M'$ are $M$-bounded sets, there is an automorphism of $M'$ fixing $A$ and mapping $B$ into $M$.

When $\bar{a}$ and $\bar{b}$ are in a monster model $M$, we have that $\text{tp}^g(\bar{a}/\emptyset, M) = \text{tp}^g(\bar{b}/\emptyset, M)$ if and only if there is $f \in \text{Aut}(M)$ mapping $\bar{a}$ to $\bar{b}$. Also for an arbitrary set $A$ we write that $\text{tp}^g(\bar{a}/A, M) = \text{tp}^g(\bar{b}/A, M)$ if and only if there is $f \in \text{Aut}(M/A)$ mapping $\bar{a}$ to $\bar{b}$, where

$$\text{Aut}(M/A) = \{f \in \text{Aut}(M) : f|A \text{ is the identity}\}. $$

We define another notion of type, called the weak type, by $\text{tp}^w(\bar{a}/A, M) = \text{tp}^w(\bar{b}/A, M)$ if for each finite $A_0 \subset A$, $\text{tp}^g(\bar{a}/A_0, M) = \text{tp}^w(\bar{b}/A_0, M)$.

**Remark 2.3.** Let $A$ and sequences $I, J$ be bounded in a monster model $M$. Let also a monster model $M'$ extend $M$. Then there is $f \in \text{Aut}(M/A)$ sending $I$ to $J$ if and only if there is $g \in \text{Aut}(M'/A)$ sending $I$ to $J$.

By Remark 2.3, if $\bar{a}, \bar{b} \in M$, $A \subset M$ is $M$-bounded and $M \preceq_{\mathbb{K}} M'$ are monster models, then $\text{tp}^g(\bar{a}/A, M) = \text{tp}^g(\bar{b}/A, M)$ if and only if $\text{tp}^g(\bar{a}/A, M') = \text{tp}^g(\bar{b}/A, M')$. Since we assume that all the sets under discussion are bounded sub-sets of the monster model, we write only $\text{tp}^g(\bar{a}/A)$ for Galois type and $\text{tp}^w(\bar{a}/A)$ for weak type.

For any $M$-bounded ordinal $\alpha$, we say that a sequence $(\bar{a}_i)_{i<\alpha}$, of tuples is strongly $A$-indiscernible in $M$, if for any $M$-bounded ordinal $\beta \geq \alpha$ we can extend the sequence to $(\bar{a}_i)_{i<\beta}$ such that for any partial order-preserving $f : \beta \rightarrow \beta$ we can find $F \in \text{Aut}(M/A)$ mapping $\bar{a}_i$ to $\bar{a}_{f(i)}$ for each $i \in \text{dom}(f)$.

The proof of the following lemma is skipped, but it is proved as Proposition 2.13 in [7].

**Lemma 2.4.** (Shelah) For every $M$-bounded cardinal $\kappa$ there exists a cardinal $H(\kappa)$ such that the following holds. Whenever $A$ is a set of size $\kappa$ and $(\bar{a}_i)_{i<H(\kappa)} \subset M$ are distinct tuples, there exists a strongly $A$-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ in $M$ such that for each $n < \omega$ there are $i_0 < ... < i_n < H(\kappa)$ such that

$$\text{tp}^g(\bar{b}_0, ..., \bar{b}_n/A) = \text{tp}^g(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A).$$

Furthermore, if $I$ is any linear ordering, there exists a monster model $M'$ extending $M$ and $(\bar{a}_i)_{i \in I}$ in $M'$ such that for any $n < \omega$ and $j_0 < ... < j_n \in I$ there are $i_0 < ... < i_n < H(\kappa)$ such that

$$\text{tp}^g(\bar{b}_0, ..., \bar{b}_n/A) = \text{tp}^g(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A).$$
We denote $H(\aleph_0) = H$. We know that $H = \bigcup_{(2^{\aleph_0})^+}$, which is the so called Hanf number of abstract elementary classes with $LS(\mathbb{K}) = \aleph_0$. We will always assume that the cardinal $\mu$ related to the monster model is closed under the operation $H(\cdot)$, that is, when a set $A$ is bounded in $\mathfrak{M}$, also the cardinal $H(|A|)$ is bounded in $\mathfrak{M}$. We can find arbitrarily large such cardinals: for any $\kappa$, define $\mu_0 = \kappa$ and $\mu_{n+1} = H(\mu_n)$. When $\mu = \bigcup_{n<\omega}\mu_n$, we have that $\lambda < \mu$ implies $H(\lambda) < \mu$.

Now we see that also the notion of any finitely many tuples being included in a strongly $A$-indiscernible sequence is independent from the monster model for bounded $A$. Let $(\bar{a}_0, ..., \bar{a}_n)$ be included in some strongly $A$-indiscernible sequence $(\bar{a}_i)_{i<\alpha}$ in $\mathfrak{M}$. We can extend this sequence to the bounded length $H(|A|)$. Then in any monster model $\mathfrak{M}'$, such that $\mathfrak{M} \preceq_K \mathfrak{M}'$, there is a strongly indiscernible $(\bar{b}_i)_{i<\lambda}$ and $i_0 < ... < i_n < H(|A|)$ such that

$$tp^g(\bar{b}_0, ..., \bar{b}_n/A) = tp^g(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A) = tp^g(\bar{a}_0, ..., \bar{a}_n/A).$$

Thus we have $f \in \text{Aut}(\mathfrak{M}'/A)$ mapping $\bar{b}_k$ to $\bar{a}_k$ for each $0 \leq k \leq n$. The sequence $(f(\bar{b}_i))_{i<\lambda}$ is strongly indiscernible in the extended monster model.

Similarly, Lemma 2.4 implies that if there are more than $H(|A|)$ many distinct tuples, then for any $n < \omega$ we can find some $n$ of these tuples such that they are the beginning of a strongly $A$-indiscernible sequence.

We say that a weak type $tp^w(\bar{a}/A)$ $\text{Lascar-splits}$ over finite $E \subseteq A$ if there is a strongly $E$-indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\bar{a}_0, \bar{a}_1$ are in $A$ and $tp^w(\bar{a}_0/\bar{a} \cup E) \neq tp^w(\bar{a}_1/\bar{a} \cup E)$. The notion of Lascar-splitting is also independent of the monster model. We define our notion of independence with built-in extension property.

**Definition 2.5.** We say that $\bar{a}$ is independent of $B$ over $A$, written

$$\bar{a} \nvDash_A B,$$

if there is finite $E \subseteq A$ such that for all monster models $\mathfrak{M}'$ extending $\mathfrak{M}$ and $D \subseteq \mathfrak{M}'$ such that $A \cup B \subseteq D$ there is a monster model $\mathfrak{M}''$ extending $\mathfrak{M}'$ and $\bar{b} \in \mathfrak{M}''$ such that $t^w(\bar{b}/A \cup B) = t^w(\bar{a}/A \cup B)$ and $t^w(\bar{b}/D)$ does not $\text{Lascar-split}$ over $E$.

If $\mathfrak{M} \preceq_K \mathfrak{M}'$ are monster models and $\bar{a}, A \cup B \subseteq \mathfrak{M}$ are bounded, then $\bar{a} \nvDash_A B$ in $\mathfrak{M}$ if and only if $\bar{a} \nvDash_A B$ in $\mathfrak{M}'$. Also by $\mu$-saturation, if $D$ in the previous definition is $\mathfrak{M}$-bounded, then we can find $\bar{b}$ in $\mathfrak{M}$.

We study several monster models instead of one, since we want to be exact with our notion of boundedness. Usually the monster model is only assumed to be 'large enough' and all sets in question 'small enough', but we want to be clear with the details. The main difficulty with only one monster model would be in the proof of Proposition 2.7, where we would have to assume that a bounded union of bounded
sets is also bounded. In the other hand we want the least unbounded cardinal in \( \mathfrak{M} \) to be a limit, and hence it would have to be a regular limit cardinal. It is consistent with ZFC that such cardinals do not exist above \( \aleph_0 \).

The following properties are clear by the definition.

**Proposition 2.6.**

1. **Invariance:** Assume that \( f \) is an automorphism of \( \mathfrak{M} \), \( \bar{a}, A, B \subseteq \mathfrak{M} \) are bounded and \( \bar{a} \downarrow_A B \). Then \( f(\bar{a}) \downarrow_B f(\bar{B}) \). Also if \( \text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B) \), then \( \bar{b} \downarrow_A B \).

2. **Monotonicity:** Assume that \( A \subseteq B \subseteq C \subseteq D \) and \( \bar{a} \downarrow_A D \). Then \( \bar{a} \downarrow_B C \).

3. **Restricted local character:** Assume that \( \bar{a} \downarrow_A B \). Then there is finite \( E \subseteq A \) such that \( \bar{a} \downarrow_E (A \cup B) \).

Now we see that "built-in extension" truly gives us the extension property.

**Proposition 2.7** (Extension). Assume that \( \bar{a} \downarrow_A B \) and \( A \subseteq B \subseteq D \), (where all the sets are bounded in \( \mathfrak{M} \)). Then there exists \( \bar{a}'(\in \mathfrak{M}) \) such that \( \text{tp}^w(\bar{a}/B) = \text{tp}^w(\bar{a}'/B) \) and \( \bar{a}' \downarrow_A D \).

**Proof.** By Proposition 2.6(3), we may assume that \( A \) is finite. Enumerate all types \( \text{tp}^w(\bar{b}/D) \), \( i < \kappa \) (in \( \mathfrak{M} \)) such that \( \text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B) \) and \( \text{tp}^w(\bar{b}/D) \) does not Lascar split over \( A \) for all \( i < \kappa \). This set is nonempty, since \( \bar{a} \downarrow_A B \). For each \( i < \kappa \), if \( \bar{b}_i \not\in_A D \) let, \( E_i \subseteq \mathfrak{M}_i \) be a set witnessing this, i.e. some set in some monster-extension \( \mathfrak{M}_i \) such that \( D \subseteq E_i \) and \( \text{tp}^w(\bar{b}/D) \) does not have a non-splitting extension to \( E_i \). If \( \bar{b}_i \downarrow_A D \), let \( E_i = D \).

The set \( E = \bigcup_{i < \kappa} E_i \) is bounded in some monster-extension \( \mathfrak{M}' \). But \( \bar{a} \downarrow_A B \) also in \( \mathfrak{M}' \), and thus there is \( \bar{b} \in \mathfrak{M}' \) such that \( \text{tp}^w(\bar{b}/B) = \text{tp}^w(\bar{a}/B) \) and \( \text{tp}^w(\bar{b}/E) \) does not Lascar-split over \( A \). Now \( \text{tp}^w(\bar{b}/D) \) does not Lascar-split over \( A \) either, and by \( \mu \)-saturation, there is \( i < \kappa \) such that \( \text{tp}^w(\bar{b}/D) = \text{tp}^w(\bar{b}_i/D) \). But \( \text{tp}^w(\bar{b}/D) \) has a non-splitting extension to \( E_i \), namely \( \bar{b} \), and thus \( \bar{b}_i \downarrow_A D \), and \( \bar{b}_i \) is as we wanted.

We use both notations \( \bar{a}\bar{b} \) and \( \bar{a}^{-}\bar{b} \) for the concatenation of tuples. We can also abbreviate \( \bar{a} \cup \bar{b} \) for \( \{\bar{a}\} \cup \{\bar{b}\} \) and \( \bar{a} \in A \) for \( \bar{a} \in A^{\text{lg}(\bar{a})} \).

**Proposition 2.8** (Finite Pairs Lemma). Let \( B \) be finite and \( A \subseteq B \). Assume that \( \bar{a} \downarrow_A B \) and \( \bar{b} \downarrow_{(A,\bar{a})} B \cup \bar{a} \). Then \( \bar{b} \downarrow_A B \).

**Proof.** Assume, for a contradiction, that \( \bar{a}^{-}\bar{b} \not\in_A B \). Especially, the finite set \( A \) does not witness that \( \bar{a}^{-}\bar{b} \downarrow_A B \). Hence, there is \( D \) containing \( B \) such that whenever \( \text{tp}^w(\bar{a}',\bar{b}/B) = \text{tp}^w(\bar{a},\bar{b}/B) \), then \( \text{tp}^w(\bar{a}',\bar{b}/D) \) Lascar-splits over \( A \). We may increase the set \( D \) if necessary, and assume that it has the following property:
For every finite \( A \subset D \) and tuples \( \bar{a}_0, \bar{a}_1 \in D \) such that they are a beginning of a strongly \( A \)-indiscernible sequence \((\bar{a}_i)_{i<\omega} \), there is one such sequence in \( D \).

By definition there is \( \bar{a}' \in \mathcal{M}' \) such that \( \text{tp}^w(\bar{a}'/B) = \text{tp}^w(\bar{a}/B) \) and \( \text{tp}^w(\bar{a}'/D) \) does not Lascar-split over \( A \). Since \( B \) is finite, we have \( f \in \text{Aut}(\mathcal{M}/B) \) such that \( f(\bar{a}) = \bar{a}' \). Now \( \text{tp}^w(\bar{a}', f(\bar{b})/B) = \text{tp}^w(\bar{a}, \bar{b}/B) \), and thus \( f(\bar{b}) \downarrow_{\text{Aut}(\mathcal{M}/\bar{a}'')} B \cup \bar{a}' \).

Again by definition there is \( \bar{b} \in \mathcal{M}' \) such that \( \text{tp}^w(\bar{b}/B \cup \bar{a}') = \text{tp}^w(f(\bar{b})/B \cup \bar{a}') \) and \( \text{tp}^w(\bar{b}/D \cup \bar{a}') \) does not Lascar-split over \( (A \cup \bar{a}') \). Hence also \( \text{tp}^w(\bar{a}', \bar{b}/B) = \text{tp}^w(\bar{a}', f(\bar{b})/B) = \text{tp}^w(\bar{a}, \bar{b}/B) \).

Let \( (\bar{c}_i)_{i<\omega} \) be strongly \( A \)-indiscernible such that \( \text{tp}^g(\bar{c}_0/A \cup \bar{a}' \cup \bar{b}') \neq \text{tp}^g(\bar{c}_1/A \cup \bar{a}' \cup \bar{b}') \) and \( \bar{c}_0, \bar{c}_1 \in D \). By strong indiscernibility, this sequence extends to strongly \( A \)-indiscernible \((\bar{c}_i)_{i<\omega} \). By the above property of \( D \), we may assume that \((\bar{c}_i)_{i<\omega} \) is in \( D \). Since there are either \( H \)-many \( \bar{c}_i \) not realizing \( \text{tp}^g(\bar{c}_0/A \cup \bar{a}' \cup \bar{b}') \) or \( H \)-many \( \bar{c}_i \) not realizing \( \text{tp}^g(\bar{c}_1/A \cup \bar{a}' \cup \bar{b}') \), we may assume that

\[
\text{tp}^g(\bar{c}_0/A \cup \bar{a}' \cup \bar{b}') \neq \text{tp}^g(\bar{c}_1/A \cup \bar{a}' \cup \bar{b}')
\]

for each \( i < H \).

We claim that \((\bar{c}_i)_{i<\omega} \) has the property that for any \( i_0 < i_1 < H \),

\[
\text{tp}^w(\bar{c}_{i_0}, \bar{c}_{i_1}/A \cup \bar{a}') = \text{tp}^w(\bar{c}_0, \bar{c}_1/A \cup \bar{a}').
\]

Assume, for a contradiction, that there are \( i_0 < i_1 \) such that the above does not hold. We check the following three possibilities:

1. \( 1 < i_0 \)
2. \( i_0 = 0 \) or
3. \( i_0 = 1 \).

Assume that (1) holds. Since \( H \) is a cardinal, we may skip less than \( H \) many tuples if necessary and assume that \( i_0 = 2 \) and \( i_1 = 3 \). The sequence \((\bar{d}_i)_{i<\omega} \), where \( \bar{d}_i = (\bar{c}_{\alpha+2n}, \bar{c}_{\alpha+2n+1}) \) for \( i = \alpha + n < H \), \( \alpha \) limit and \( n < \omega \), is strongly \( A \)-indiscernible and \( \text{tp}^g(\bar{d}_0/A \cup \bar{a}') \neq \text{tp}^g(\bar{d}_1/A \cup \bar{a}') \).

Then we have that \( \text{tp}^w(\bar{a}'/D) \) Lascar-splits over \( A \), a contradiction. If we have (2), then the sequence \((\bar{c}_0, \bar{c}_i)_{i<\omega} \) is strongly \( A \)-indiscernible with \( \text{tp}^w(\bar{c}_0, \bar{c}_1/A \cup \bar{a}') \neq \text{tp}^w(\bar{c}_0, \bar{c}_1/A \cup \bar{a}') \). We get again that \( \text{tp}^w(\bar{a}'/D) \) Lascar-splits over \( A \), a contradiction.

Assume that (1) or (2) does not hold, and thus for all indexes \( i_0 < i_1 \) such that \( \text{tp}^w(\bar{c}_{i_0}, \bar{c}_{i_1}/A \cup \bar{a}') \neq \text{tp}^w(\bar{c}_0, \bar{c}_1/A \cup \bar{a}') \), we have \( i_0 = 1 \). We can study the strongly \( A \)-indiscernible sequence \((\bar{c}_i)_{i<\omega, i \neq 1} \), since \( \text{tp}^g(\bar{c}_0/A \cup \bar{a}' \cup \bar{b}') \neq \text{tp}^g(\bar{c}_2/A \cup \bar{a}' \cup \bar{b}') \) and \( \bar{c}_0, \bar{c}_2 \in D \). The claim holds for this sequence.

We have shown the claim. Now by Lemma 2.4, there is a strongly \((A \cup \bar{a}')\)-indiscernible sequence \((\bar{c}_i')_{i<\omega} \) such that \( \text{tp}(\bar{c}_0', \bar{c}_1'/A \cup \bar{a}') = \text{tp}(\bar{c}_{i_0}, \bar{c}_{i_1}/A \cup \bar{a}) \) for some \( i_0 < i_1 < H \). By the previous claim we have \( f \in \text{Aut}(\mathcal{M}'/A \cup \bar{a}') \) mapping \((\bar{c}_0, \bar{c}_1') \) to...
and $\bar{c}_0, \bar{c}_1$) and thus may assume that $\bar{c}_0 = \bar{c}_0$ and $\bar{c}_1 = \bar{c}_1$. Since $\text{tp}(\bar{c}_0/(A \cup \bar{a}') \cup \bar{b}') \neq \text{tp}(\bar{c}_1/(A \cup \bar{a}) \cup \bar{b})$, we now have that $\text{tp}(\bar{b}/D \cup \bar{a}')$ Lascar-splits over $(A \cup \bar{a}')$, a contradiction. 

Let $A \subset \mathcal{M}$ be a bounded subset and $\bar{a} \in \mathcal{M}$ a tuple. We say that $\text{tp}(\bar{a}/A)$ is \textit{bounded}, if the set $\{b \in \mathcal{M} : b \models \text{tp}(\bar{a}/A)\}$ is bounded in $\mathcal{M}$. We see that if $\text{tp}(\bar{a}/A)$ is bounded, 

$$\{b \in \mathcal{M} : b \models \text{tp}(\bar{a}/A)\} < H(|A|).$$

Namely, if the set would have some bounded size $\kappa \geq H(|A|)$, we could find a strongly $A$-indiscernible sequence of distinct tuples realizing $\text{tp}(\bar{a}/A)$. Hence by strong indiscernibility, there would be at least $\kappa^+$-many tuples in $\mathcal{M}$ realizing $\text{tp}(\bar{a}/A)$. This is a contradiction. Since in any monster model $H(|A|)$ is bounded if and only if $A$ is bounded, a type $\text{tp}(\bar{a}/A)$ is bounded in $\mathcal{M}$ if and only if it is bounded in all extending monster models. Also by $\mu$-saturation, if $A$ is bounded in $\mathcal{M}$, $\text{tp}(\bar{a}/A)$ is bounded and $\mathcal{M}'$ is a monster model extending $\mathcal{M}$, then $\{b \in \mathcal{M} : b \models \text{tp}(\bar{a}/A)\} = \{b \in \mathcal{M}' : b \models \text{tp}(\bar{a}/A)\}$.

\textbf{Lemma 2.9.} Let $A$ be finite.

(1) If $\text{tp}(\bar{a}/A)$ is bounded, then $\bar{a} \downarrow_A B$ for any $B$.
(2) If $\text{tp}(\bar{a}/A)$ is not bounded, then $\bar{a} \not\downarrow_A \bar{a}$.

\textbf{Proof.} Let $\mathcal{M}'$ be any monster-extension and $D \subset \mathcal{M}'$ any set. Assume that $\text{tp}(\bar{a}/B)$ does split over $A$. Let $(\bar{b}_i)_{i < H}$ be strongly $A$-indiscernible such that $\text{tp}(\bar{b}_i/A \cup \{\bar{a}\}) \neq \text{tp}(\bar{b}_i/A \cup \{\bar{a}\})$. There has to be either $H$-many $i$ such that $\text{tp}(\bar{b}_i/A \cup \{\bar{a}\}) \neq \text{tp}(\bar{b}_i/A \cup \{\bar{a}\})$ or $H$-many such $i$ that $\text{tp}(\bar{b}_i/A \cup \{\bar{a}\}) \neq \text{tp}(\bar{b}_i/A \cup \{\bar{a}\})$. Thus we may assume that $\text{tp}(\bar{a}/A \cup \{\bar{a}\}) \neq \text{tp}(\bar{b}_0/A \cup \{\bar{a}\})$ for all $0 < i < H$. By strong $A$-indiscernibility, for each $i < H$, there is $f_i \in \text{Aut}(\mathcal{M}'/A)$ such that $f_i(\bar{b}_k) = \bar{b}_{i+k}$ for all $k < H$. Now if $i < j$ we have that $f_i(\bar{a}) \neq f_j(\bar{a})$. Otherwise we would have that $(f_i - 1 \circ f_j)(\bar{a}) = \bar{a}$ and $(f_i - 1 \circ f_j)(\bar{b}_0) = \bar{b}_k$ for $k > 0$. Now $(f_i(\bar{a}))_{i < H}$ are different realizations of $\text{tp}(\bar{a}/A)$, and the type $\text{tp}(\bar{a}/A)$ is not bounded. Thus if $\text{tp}(\bar{a}/A)$ is bounded, $\text{tp}(\bar{a}/D)$ does not split over $A$ for any $D$ or monster-extension $\mathcal{M}'$. This proves (1).

To prove (2), assume that $\text{tp}(\bar{a}/A)$ is not bounded. There are $H$-many tupples $\bar{b}$, such that $\text{tp}(\bar{b}/A) = \text{tp}(\bar{a}/A)$. By Lemma 2.4, there is a strongly $A$-indiscernible sequence $(\bar{a}_i)_{i < \omega}$ of distinct tuples such that $\text{tp}(\bar{a}_0/A) = \text{tp}(\bar{a}/A)$ and hence $\text{tp}(\bar{a}_i/A) = \text{tp}(\bar{a}/A)$ for each $i < \omega$. Furthermore, since we have $f \in \text{Aut}(\mathcal{M}/A)$ mapping $\bar{a}_0$ to $\bar{a}$, we may assume that $a_0 = \bar{a}$. Assume, for a contradiction, that $\bar{a} \downarrow_A \bar{a}$. Then let $\bar{a}'$ be such that $\text{tp}(\bar{a}'/A \cup \{\bar{a}\}) = \text{tp}(\bar{a}/E \cup \{\bar{a}\})$ and $\text{tp}(\bar{a}'/A \cup \{\bar{a}_i : i < \omega\})$ does not Lascar-split over $A$. But now we must have
that \( \bar{a}' = \bar{a} \) and this is a contradiction, since \( \text{tp}^\emptyset(\bar{a}_0/A \cup \{\bar{a}\}) \neq \text{tp}^\emptyset(\bar{a}_1/A \cup \{\bar{a}\}) \) and thus \( \text{tp}^w(\bar{a}/A \cup \{\bar{a}_i : i < \omega\}) \) does Lascar-split over \( A \). This proves (2).

**Proposition 2.10.** Let \( A \subset B \) be finite, \( \bar{a} \downarrow_A B \) and \( B \subset D \). There is \( \bar{a}' \) such that \( (\bar{a}, \bar{a}') \) is a beginning of a strongly \( B \)-indiscernible sequence and \( \bar{a}' \downarrow_A D \).

**Proof.** If \( \text{tp}^w(\bar{a}/A) \) is bounded, we can take the constant sequence, which is strongly \( A \)-indiscernible, by Lemma 2.9(1). We assume that \( \text{tp}^w(\bar{a}/A) \) is unbounded. By extension there are \( \bar{a}_i \), for \( i < H \) such that \( \text{tp}^w(\bar{a}_i/B) = \text{tp}^w(\bar{a}/B) \) and \( \bar{a}_i \downarrow_A B \cup \bigcup_{j<i}\{\bar{a}_j\} \). By Lemma 2.9(2) and monotonicity we have that \( \bar{a}_j \neq \bar{a}_i \) for any \( j \neq i \). Thus we have \( j_0, j_1 < H \) such that \( (\bar{a}_{j_0}, \bar{a}_{j_1}) \) is a beginning of a strongly \( B \)-indiscernible sequence. Since \( B \) is finite, there is \( f \in \text{Aut}(M/B) \) mapping \( \bar{a}_{j_0} \) to \( \bar{a} \). Denote \( \bar{a}^* = f(\bar{a}_{j_1}) \). Now \( (\bar{a}, \bar{a}^*) \) is the beginning of a strongly \( B \)-indiscernible sequence and \( \bar{a}^* \downarrow_A B \cup \{\bar{a}\} \). Again by extension there is \( \bar{a}' \) such that \( \bar{a}' \downarrow_A D \) and \( \text{tp}^w(\bar{a}'/B \cup \{\bar{a}\}) = \text{tp}^w(\bar{a}^*/B \cup \{\bar{a}\}) \). Let \( g \in \text{Aut}(B \cup \{\bar{a}\}) \) be such that \( g(\bar{a}^*) = \bar{a}' \). Then also \( (g(\bar{a}), g(\bar{a}^*)) = (\bar{a}, \bar{a}') \) is a beginning of a strongly \( B \)-indiscernible sequence. \( \square \)

### 2.1. Lascar strong types.

We say that an equivalence relation \( E \) in a monster model \( M \) is \( A \)-invariant, if it is preserved by each \( f \in \text{Aut}(M/A) \). We also say that an equivalence relation \( E \) is bounded, if the number of equivalence classes of \( E \) in \( M \) is bounded. If \( E \) is a bounded and \( A \)-invariant equivalence relation and \( (\bar{a}_i)_{i<\omega} \) a strongly \( A \)-indiscernible sequence, then \( E(\bar{a}_{i_0}, \bar{a}_{i_1}) \) for each \( i_0, i_1 < \omega \). Otherwise we would get due to \( A \)-invariance that \( \neg E(\bar{a}_i, \bar{a}_j) \) for each \( \bar{a}_i, \bar{a}_j \) in the sequence.

If the number of equivalence classes of \( E \) is \( \kappa \), we could extend the sequence to the length of \( \kappa^+ \), and get a contradiction.

We conclude that if an \( A \)-invariant equivalence relation has a bounded number of equivalence classes in \( M \), the number must be strictly less than \( H(|A|) \). Also by \( \mu \)-saturation it can not have any other equivalence classes in any extending monster model.

**Definition 2.11 (Lascar strong type).** We say that \( \bar{a} \) and \( \bar{b} \) have the same **Lascar strong type** over \( A \), written

\[
\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A),
\]

if \( \ell(\bar{a}) = \ell(\bar{b}) \) and \( E(\bar{a}, \bar{b}) \) holds for any \( A \)-invariant and bounded equivalence relation \( E \) of \( \ell(\bar{a}) \)-tuples.

Each tuple in a strongly \( A \)-indiscernible sequence has the same Lascar strong type over \( A \). Thus the number of Lascar strong types over a set \( A \) is strictly smaller than the cardinal \( H(|A|) \). This holds by by Lemma 2.4, since for any sequence \( (\bar{a}_i)_{i<\kappa} \) for \( \kappa \geq H(A) \), there are indexes \( i < j < \kappa \), such that the tuples \( \bar{a}_i, \bar{a}_j \) are in the
same strongly $A$-indiscernible sequence. As a corollary of Proposition 2.10 we get the following.

**Corollary 2.12.** Let $A \subset B$ and $\bar{a} \downarrow_A B$, where $B$ is finite. Then there is $\bar{a}'$ such that $\text{Lstp}(\bar{a}'/B) = \text{Lstp}(\bar{a}/B)$ and $\bar{a}' \downarrow_A D$.

We give an equivalent condition for two tuples to have the same Lascar strong type over $A$.

**Proposition 2.13.** The following are equivalent.

1. $\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)$.
2. There exists $n < \omega$, $\bar{a}_i$ for $i \leq n$ and strongly $A$-indiscernible sequences $J_i$ for $i < n$ such that $\bar{a}_0 = \bar{a}$, $\bar{a}_n = \bar{b}$ and $\bar{a}_i, \bar{a}_{i+1} \in J_i$ for $i < n$.

*Proof.* Since elements in a strongly $A$-indiscernible sequence have same Lascar strong types over $A$, (2) implies (1). We show that (1) implies (2). It is enough to show that the relation defined by (2) is an $A$-invariant equivalence relation with a bounded number of classes. It is clearly $A$-invariant, transitive and symmetric. The trivial strongly $A$-indiscernible sequence $\langle \bar{a}_i \rangle_{i < \omega}$ shows that it is also reflexive. We are left to show that it is bounded. Assume that it would not be bounded, and thus there would be $H(|A|)$-many inequivalent tuples. But by Lemma 2.4, at least two of these elements would be included in some strongly $A$-indiscernible sequence, a contradiction. $\square$

At least by the previous proposition it is clear that the relation $\text{Lstp}(\bar{a}/A)$ does not depend on the possible extension of the monster model.

If $\langle \bar{a}_i \rangle_{i < \alpha}$ is a strongly $E \cup \bar{c}$-indiscernible sequence, the sequence $\langle \bar{a}_i \bar{c} \rangle_{i < \alpha}$ is strongly $E$-indiscernible. The previous Proposition implies that

$$\text{Lstp}(\bar{a}/E \cup \bar{c}) = \text{Lstp}(\bar{b}/E \cup \bar{c}) \Rightarrow \text{Lstp}(\bar{a} \bar{c}/E) = \text{Lstp}(\bar{b} \bar{c}/E).$$

**Definition 2.14** (Strong automorphism). We say that $f \in \text{Aut}(\mathfrak{M}/A)$ is a strong automorphism over $A$ if $\text{Lstp}(\bar{a}/A) = \text{Lstp}(f(\bar{a})/A)$ for each tuple $\bar{a}$.

We define $\text{Saut}(\mathfrak{M}/A)$ to be the group of strong automorphisms fixing $A$ pointwise. The group $\text{Saut}(\mathfrak{M}/A)$ is a normal subgroup of the automorphism group $\text{Aut}(\mathfrak{M}/A)$.

**Proposition 2.15.** The following are equivalent for a bounded $A$.

1. $\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)$.
2. There is $f \in \text{Saut}(\mathfrak{M}/A)$ such that $f(\bar{a}) = \bar{b}$.
Proof. By the definition of a strong automorphism, (2) implies (1). To prove that (1) implies (2), we show that the equivalence relation defined by (2) is $A$-invariant and has a bounded number of equivalence classes. First, it is $A$-invariant due to the normality of the subgroup $\text{Saut}(\mathcal{M}/A)$ of $\text{Aut}(\mathcal{M}/A)$. To prove that it is bounded, assume the contrary that $(\bar{a}_i)_{i<\text{H}(\mathcal{M}/A)}$ are distinct tuples. We remark that the cardinal $\text{H}(\mathcal{M}/A)$ is bounded. Let $\mathcal{A}$ be a model of size $\text{H}(\mathcal{M}/A)$ such that $A \subset \mathcal{A}$ and each Lascar strong type over $A$ is represented in $\mathcal{A}$. By Lemma 2.4 there are $i_0 < i_1 < \text{H}(\mathcal{M}/A) = \text{H}(\mathcal{A})$ such that $(\bar{a}_{i_0}, \bar{a}_{i_1})$ is the beginning of a strongly $\mathcal{A}$-indiscernible sequence. Thus there is $f \in \text{Aut}(\mathcal{M}/\mathcal{A})$ mapping $\bar{a}_{i_0}$ to $\bar{a}_{i_1}$. We show that this automorphism is actually strongly over $\mathcal{A}$, which implies that $\bar{a}_{i_0}$ and $\bar{a}_{i_1}$ are equivalent. For this, let $\bar{a} \in \mathcal{M}$ be arbitrary. There is $\bar{a}' \in \mathcal{A}$ realizing $\text{Lstp}(\bar{a}/A)$. Since $\text{Lstp}$ is an $A$-invariant notion and $f(\bar{a}') = \bar{a}'$, we conclude that

$$\text{Lstp}(f(\bar{a})/A) = \text{Lstp}(f(\bar{a}')/A) = \text{Lstp}(\bar{a}'/A) = \text{Lstp}(\bar{a}/A).$$

\[\square\]

The previous equivalence implies that if $\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)$ and $\bar{c}$ is an arbitrary tuple, we can always find such $\bar{d}$ that $\text{Lstp}(\bar{a} \bar{d}/A) = \text{Lstp}(\bar{b} \bar{c}/A)$.

Lemma 2.16. Assume that $\mathcal{A}$ is an $\aleph_0$-saturated model. Then the following are equivalent.

1. $\text{tp}^w(\bar{a}/\mathcal{A})$ Lascar-splits over finite $E \subset \mathcal{A}$.
2. There are tuples $\bar{c}, \bar{d} \in \mathcal{A}$ such that $\text{Lstp}(\bar{c}/E) = \text{Lstp}(\bar{d}/E)$ but $\text{tp}^g(\bar{c}/E \cup \bar{a}) \neq \text{tp}^g(\bar{d}/E \cup \bar{a})$.

Proof. Item (1) implies (2) by the definition. We show that if (1) does not hold, then neither does (2). For this, assume that $\text{tp}^w(\bar{a}/\mathcal{A})$ does not Lascar-split over $E$ and $\bar{c}, \bar{d}$ are distinct tuples in $\mathcal{A}$ such that $\text{Lstp}(\bar{c}/E) = \text{Lstp}(\bar{d}/E)$. By Proposition 2.13 there are strongly $E$-indiscernible sequences $I_k$ and tuples $(\bar{a}_k, \bar{a}_{k+1}) \in I_k$ for $0 \leq k \leq n$ such that $\bar{a}_0 = \bar{c}$ and $\bar{a}_{n+1} = \bar{d}$. Since $\mathcal{A}$ is $\aleph_0$-saturated, we may assume that each $\bar{a}_k$ is in $\mathcal{A}$. But now since $\text{tp}^w(\bar{a}/E)$ does not Lascar-split over $E$, we must have that $\text{tp}^g(\bar{a}_0/E \cup \bar{a}) = \text{tp}^g(\bar{a}_1/E \cup \bar{a}) = \ldots = \text{tp}^g(\bar{a}_n/E \cup \bar{a}) = \text{tp}^g(\bar{a}_{n+1}/E \cup \bar{a})$. \[\square\]

We define that a model $\mathcal{A}$ is a-saturated if each Lascar strong type over a finite subset of $\mathcal{A}$ is realized in $\mathcal{A}$.

2.2. Restricted properties with simplicity and weak stability. We introduce new properties called weak stability and simplicity. We say that $(\mathcal{K}, \leq_{\mathcal{K}})$ is weakly stable in a cardinal $\lambda$, if whenever $|A| \leq \lambda$ and $(\bar{a}_i)_{i<\lambda^+}$ are tuples, there are $i < j < \lambda^+$ such that $\text{tp}^w(\bar{a}_i/A) = \text{tp}^w(\bar{a}_j/A)$. 

**Definition 2.17** (Weak stability). We say that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is weakly stable if there is a cardinal \(\lambda\) such that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is weakly stable in \(\lambda\).

**Definition 2.18** (Simplicity). We say that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is simple, if \(\bar{a} \downarrow_A A\) for each tuple \(\bar{a}\) and finite set \(A\).

In the \(\aleph_0\)-stable case in [7] we defined simplicity as the assumption that for any \(\bar{a}\) and arbitrary \(A\) there is finite \(A' \subset A\) such that \(\bar{a} \downarrow_{A'} A\). Here we call this property *local character*. With \(\aleph_0\)-stability, the above notion of simplicity is equivalent with local character\(^2\).

In this section we collect those properties of the notion \(\downarrow\) which we can derive from these restricted versions of simplicity and stability. From now on we will always assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is simple and weakly stable.

Weak stability and simplicity are needed to prove finite symmetry for \(\downarrow\). Then we will use simplicity and finite symmetry to prove several other properties.

**Proposition 2.19.** Assume that \(A\) is finite, \(\bar{a} \downarrow_A \bar{b}\) and \(\bar{b} \not\equiv_A \bar{a}\). Then there are \(\bar{a}_i, \bar{b}_i\) for \(i < H\) such that \(\bar{b}_i \downarrow_A \bar{a}_j\) if and only if \(i > j\).

*Proof.* Let \(\bar{a}_0 = \bar{a}\) and \(\bar{b}_0 = \bar{b}\). Define \(\bar{a}_i, \bar{b}_i\) by induction such that

1. \(\text{tp}^u(\bar{a}_i \bar{b}_i/A) = \text{tp}^u(\bar{a} \bar{b}/A)\) for all \(i < H\).
2. \(\text{tp}^u(\bar{a}_i/A \bar{b}) = \text{tp}^u(\bar{a}/A \bar{b})\) for all \(i < H\).
3. The pair \((\bar{b}, \bar{b}_i)\) is a beginning of a strongly \(A\)-indiscernible sequence for each \(0 < i < H\).
4. \(\bar{a}_i \bar{b}_i \downarrow_A B\) for every finite \(B \subset \bigcup_{j<i} \{\bar{a}_j, \bar{b}_j\}\).

Note that due to simplicity, item 4 holds also when \(i = 0\). Assume we have defined \(\bar{a}_i, \bar{b}_i\) for \(i < \alpha\). Since \(\bar{a} \downarrow_A \bar{b}\), we get by extension \(\bar{a}_\alpha\) such that \(\text{tp}^u(\bar{a}_\alpha/A \bar{b}) = \text{tp}^u(\bar{a}/A \bar{b})\) and \(\bar{a}_\alpha \downarrow_A \bigcup_{i<\alpha} \{\bar{a}_i, \bar{b}_i\}\). By simplicity, \(\bar{b} \downarrow_{(A \cup \bar{a}_\alpha)} A \cup \bar{a}_\alpha\), and then by Proposition 2.10 we find \(\bar{b}_\alpha\) such that \((\bar{b}, \bar{b}_\alpha)\) is a beginning of a strongly \((A \cup \bar{a}_\alpha)-\)indiscernible sequence and \(\bar{b}_\alpha \downarrow_{(A \cup \bar{a}_\alpha)} \bigcup_{i<\alpha} A \cup \{\bar{a}_\alpha\} \cup \{\bar{a}_i, \bar{b}_i\}\). Now both (2) and (3) hold for \(i \leq \alpha\).

Let \(f \in \text{Aut}(\mathfrak{M}/A \bar{b})\) be such that \(f(\bar{a}) = \bar{a}_\alpha\) and \(g \in \text{Aut}(\mathfrak{M}/A \bar{a}_\alpha)\) such that \(g(\bar{b}) = \bar{b}_\alpha\). Now \(g \circ f(\bar{a} \bar{b}) = \bar{a}_\alpha \bar{b}_\alpha\), and thus (1) holds. Then let \(B \subset \bigcup_{i<\alpha} \{\bar{a}_i, \bar{b}_i\}\) be finite. By monotonicity, we have that \(\bar{a}_\alpha \downarrow_A B\) and \(\bar{b}_\alpha \downarrow_{(A \cup \bar{a}_\alpha)} B\), and by Finite Pairs Lemma we get that \(\bar{a}_\alpha \bar{b}_\alpha \downarrow_A B\). Thus also (4) holds.

Finally we see that \(\bar{b}_i \downarrow_A \bar{a}_j\) if and only if \(i > j\). The case \(i = j\) follows from (1) and the assumption. Also if \(i > j\), from (4) it follows that \(\bar{b}_i \downarrow_A \bar{a}_j\). It is left to

\(^2\)The equivalence follows from Corollaries 3.28 and 3.15. In the \(\aleph_0\)-stable case we always have the Tarski-Vaught-property (Remark 3.10).
study the case when \( i < j \). Let \( 0 \leq i < j \). By item (4), \( \text{tp}^w(\bar{a}_j/A \cup \bar{b} \cup \bar{b}_i) \) does not Lascar-split over \( A \). Since \((\bar{b}, \bar{b}_i)\) is a beginning of a strongly \( A \)-indiscernible sequence, we must have that \( \text{tp}^w(\bar{b}/A \cup \bar{a}_j) = \text{tp}^w(\bar{b}_i/A \cup \bar{a}_j) \). Furthermore by item (2), \( \bar{b} \not\models_A \bar{a}_j \), and hence \( \bar{b}_i \not\models_A \bar{a}_j \).

Finally we get symmetry as in the \( \aleph_0 \)-stable case, with a suitable linear ordering contradicting weak stability.

**Proposition 2.20** (Finite symmetry). Let \( A \) be finite. Then \( \bar{a} \downarrow_A \bar{b} \) if and only if \( \bar{b} \downarrow_A \bar{a} \).

**Proof.** Assume to the contrary, that \( \bar{a} \downarrow_A \bar{b} \) but \( \bar{b} \not\models_A \bar{a} \) for some \( \bar{a}, \bar{b} \) and finite \( A \). By the previous proposition, there is a sequence \((\bar{a}_i, \bar{b}_i)_{i<\omega} \) such that \( \bar{b}_i \downarrow_A \bar{a}_j \) if and only if \( i > j \). Let \( \lambda \) be a cardinal such that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is stable in \( \lambda \). Then let \( I \) be a linear ordering such that \( |I| > \lambda \), and there is a dense set \( I_0 \subset I \) of size \( \lambda \).

By Lemma 2.4, there are \((\bar{a}_i, \bar{b}_i)_{i \in I} \) such that \( \bar{b}_i \downarrow_A \bar{a}_j \) if and only if \( i > j \). But now there are \( |I| \)-many different types over the set \((\bar{b}_i, \bar{a}_i)_{i \in I_0} \), a contradiction. \( \square \)

We continue to prove other restricted properties of \( \downarrow \).

**Lemma 2.21.** Let \( E \) be finite and \( \bar{c} \downarrow_E \bar{a} \bar{b} \). If \( \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \), then \( \text{tp}^w(\bar{a}/E \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/E \cup \{\bar{c}\}) \).

**Proof.** Let \( \mathfrak{A} \) be an \( \omega \)-saturated model containing \( E \cup \{\bar{a}, \bar{b}\} \). Let \( \bar{c}' \) be such that \( \text{tp}^w(\bar{c}'/E \cup \{\bar{a}, \bar{b}\}) = \text{tp}^w(\bar{c}/E \cup \{\bar{a}, \bar{b}\}) \) and \( \text{tp}^w(\bar{c}'/\mathfrak{A}) \) does not Lascar-split over \( E \). Let \( f \in \text{Aut}(\mathfrak{M}/E) \) be such that \( f(\bar{c}') = \bar{c} \). Hence, \( \text{tp}^w(\bar{c}/f(\mathfrak{A})) \) does not Lascar-split over \( E \). Since \( f(\mathfrak{A}) \) is an \( \omega \)-saturated model containing \( E \cup \{\bar{a}, \bar{b}\} \), and \( \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \), we must have \( \text{tp}^w(\bar{a}/E \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/E \cup \{\bar{c}\}) \) by Proposition 2.16. \( \square \)

**Lemma 2.22.** Assume that \( E \) is finite. If \( \bar{a} \downarrow_E \bar{c}, \bar{b} \downarrow_E \bar{c} \) and \( \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \), then \( \text{tp}^w(\bar{a}/E \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/E \cup \{\bar{c}\}) \).

**Proof.** Since \( \bar{a} \downarrow_E \bar{c} \), we get from Corollary 2.12 some \( \bar{a}' \) such that \( \text{Lstp}(\bar{a}'/E \cup \{\bar{c}\}) = \text{Lstp}(\bar{a}/E \cup \{\bar{c}\}) \) and \( \bar{a}' \downarrow_E \{\bar{b}, \bar{c}\} \). By the finite pairs lemma (Proposition 2.8) since \( \bar{b} \downarrow_E \bar{c} \), we must have \( (\bar{a}')^E \downarrow_E \bar{c} \). Then, by Symmetry we have that \( \bar{c} \downarrow_E \{\bar{b}, \bar{a}'\} \). But \( \text{Lstp}(\bar{a}'/E) = \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \), so \( \text{tp}^w(\bar{a}'/E \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/E \cup \{\bar{c}\}) \) by the previous Lemma. By the choice of \( \bar{a}' \), \( \text{tp}^w(\bar{a}/E \cup \{\bar{c}\}) = \text{tp}^w(\bar{b}/E \cup \{\bar{c}\}) \). \( \square \)

**Proposition 2.23** (Restricted finite character). Let \( E \) be finite. If \( \bar{a} \not\models_E B \), then there is finite \( \bar{b} \in B \) such that \( \bar{a} \not\models_E \bar{b} \).
Proof. By simplicity, we have that \( \bar{a} \downarrow_E E \). From Corollary 2.12 we get \( \bar{a}' \) such that \( \text{Lstp}(\bar{a}'/E) = \text{Lstp}(\bar{a}/E) \) and \( \bar{a}' \downarrow_E B \). Now we can’t have \( \text{tp}^w(\bar{a}'/E \cup B) = \text{tp}^w(\bar{a}/E \cup B) \), and thus there is some finite \( \bar{b} \in B \) such that \( \text{tp}^w(\bar{a}'/E \cup \{\bar{b}\}) \neq \text{tp}^w(\bar{a}/E \cup \{\bar{b}\}) \). By monotonicity, \( \bar{a}' \downarrow_E \bar{b} \). Thus we get by Lemma 2.22 that \( \bar{a} \not\subseteq_E \bar{b} \).

As a corollary we get the following.

**Lemma 2.24** (Pairs Lemma). Let \( A \subset B \). Assume that \( \bar{a} \downarrow_A B \) and \( \bar{b} \downarrow_{(A \cup \{\bar{a}\})} B \cup \{\bar{a}\} \). Then \( \bar{a} \sim \bar{b} \downarrow_A B \).

*Proof.* By Proposition 2.6(3), there is finite \( A' \subset A \) such that \( \bar{a} \downarrow_{A'} B \) and \( \bar{b} \downarrow_{(A' \cup \{\bar{a}\})} B \cup \{\bar{a}\} \). We need to show that \( \bar{a} \sim \bar{b} \downarrow_{A'} B \). But by the finite Pairs lemma, \( \bar{a} \sim \bar{b} \downarrow_{A'} B' \) for each finite \( B' \subset B \), and thus the claim follows from Restricted finite character.

Also the following proposition is clear by symmetry, monotonicity and restricted finite character.

**Proposition 2.25** (Left transitivity). Assume that \( A, B \) are finite and \( \bar{a} \cup B \downarrow_A C \). Then \( \bar{a} \downarrow_{A \cup B} C \).

**Proposition 2.26** (Transitivity). Let \( A \subset B \subset C \). If \( \bar{a} \downarrow_A B \) and \( \bar{a} \downarrow_B C \), then \( \bar{a} \downarrow_A C \).

*Proof.* By Proposition 2.6(3), there are finite \( A' \subset A \) and finite \( B' \subset B \) such that \( \bar{a} \downarrow_{A'} B \) and \( \bar{a} \downarrow_{(A' \cup B')} C \). It is enough to show that \( \bar{a} \downarrow_{A'} C \). By Proposition 2.23, it is enough to show that \( \bar{a} \downarrow_{A'} \bar{c} \) for each finite \( \bar{c} \in C \), and by finite symmetry, it is enough to show that \( \bar{c} \downarrow_{A'} \bar{a} \) for each finite \( \bar{c} \in C \). Let \( \bar{c} \in C \) be finite. Write \( \bar{c} = \bar{b} \bar{c}_0 \), where \( \bar{b} \in B \) and \( \bar{c}_0 \in C \setminus B \). We may assume that \( \bar{b} \) contains \( B' \). Then we have that \( \bar{a} \downarrow_{A'} \bar{b} \) as well as \( \bar{a} \downarrow_{(A' \cup \{\bar{b}\})} \bar{c}_0 \), and furthermore \( \bar{b} \downarrow_{A'} \bar{a} \) and \( \bar{c}_0 \downarrow_{(A' \cup \{\bar{b}\})} \bar{a} \) by symmetry. Hence, \( \bar{c} \downarrow_{A'} \bar{a} \) by the Pairs Lemma. This completes the proof for transitivity.

**Proposition 2.27** (Stationarity of Lascar strong types version 1). Let \( E \) be finite, \( \bar{a} \downarrow_E B \), \( \bar{b} \downarrow_E B \) and \( \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \). Then \( \text{tp}^w(\bar{a}/E \cup B) = \text{tp}^w(\bar{b}/E \cup B) \).

*Proof.* If not, there is finite \( \bar{c} \in B \) such that \( \text{tp}^w(\bar{a}/E \cup \bar{c}) \neq \text{tp}^w(\bar{b}/E \cup \bar{c}) \), which contradicts Lemma 2.22.

**Proposition 2.28.** Let \( A \subset B \), \( B \) finite, \( \text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A) \), \( \bar{a} \downarrow_A B \) and \( \bar{b} \downarrow_A B \). Then \( \text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{b}/B) \).
Proof. By Proposition 2.12 there is \( \bar{c} \) realizing \( \text{Lstp}(\bar{a}/B) \) such that \( \bar{c} \downarrow_A B \cup \bar{a} \cup \bar{b} \). By monotonicity and symmetry, \( \bar{a} \downarrow_B \bar{c} \) and then by transitivity, \( \bar{a} \downarrow_A B \cup \bar{c} \). Similarly, \( \bar{b} \downarrow_A B \cup \bar{c} \). Now by Proposition 2.30, \( \text{tp}^u(\bar{a}/B \cup \bar{c}) = \text{tp}^u(\bar{b}/B \cup \bar{c}) \). Since \( B \cup \bar{c} \) is finite, there is \( f \in \text{Aut}(\mathfrak{M}/B \cup \bar{c}) \) mapping \( \bar{a} \) to \( \bar{b} \). Now \( \text{Lstp}(\bar{a}/B \cup \bar{c}) = \text{Lstp}(\bar{b}/B \cup \bar{c}) \), and thus \( \text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{b}/B) \). \( \square \)

Definition 2.29 (Weak Lascar strong type). We say that \( \bar{a} \) and \( \bar{b} \) have the same weak Lascar strong type over \( A \), denote \( \text{Lstp}^w(\bar{a}/A) = \text{Lstp}^w(\bar{b}/A) \), if for every finite \( B \subset A \) we have that \( \text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{b}/B) \).

We get from the previous proposition and monotonicity the following.

Proposition 2.30 (Stationarity of Lascar strong types version 2). Let \( E \) be finite, \( \bar{a} \downarrow_E B \), \( \bar{b} \downarrow_E B \) and \( \text{Lstp}(\bar{a}/E) = \text{Lstp}(\bar{b}/E) \). Then \( \text{Lstp}^w(\bar{a}/E \cup B) = \text{Lstp}^w(\bar{b}/E \cup B) \).

We state a lemma about building Morley-type indiscernible sequences.

Lemma 2.31. Assume that \( A \) is finite and \( \bar{a} \) a tuple. For any ordinal \( \lambda \) there exists a strongly \( A \)-indiscernible sequence \( (\bar{a}_i)_{i<\lambda} \) such that \( \bar{a}_0 = \bar{a} \) and

\[
\bar{a}_i \downarrow_A \bigcup_{j<i} \bar{a}_j \text{ for each } i < \lambda.
\]

Proof. By strong indiscernibility and finite character it is enough to find a sequence for \( \lambda = \omega \).

Again if \( \text{tp}^u(\bar{a}/A) \) is bounded, we can take the trivial sequence by Lemma 2.9(1). Assume that \( \text{tp}^u(\bar{a}/A) \) is unbounded. Using simplicity and extension we define \( (\bar{b}_i)_{i<H} \) such that each \( \bar{b}_i \) realizes \( \text{tp}^g(\bar{a}/A) \) and

\[
\bar{b}_i \downarrow_A \bigcup_{j<i} \bar{b}_j \text{ for each } i < H.
\]

By Lemma 2.9(2) these \( \bar{b}_i \) are distinct and we can use Proposition 2.4 to find strongly \( A \)-indiscernible sequence \( (\bar{a}_i)_{i<\omega} \) such that for each \( n < \omega \) there are \( i_0 < ... < i_n < H \) such that

\[
\text{tp}^g(\bar{a}_0, ..., \bar{a}_n/A) = \text{tp}^g(\bar{b}_{i_0}, ..., \bar{b}_{i_n}/A).
\]

Hence \( \bar{a}_n \downarrow_A \bigcup_{i<n} \bar{a}_i \) for each \( n < \omega \). Also \( \bar{a}_0 \) realizes \( \text{tp}^g(\bar{a}/A) \) and thus there is an automorphism \( f \in \text{Aut}(\mathfrak{M}/A) \) mapping \( \bar{a}_0 \) to \( \bar{a} \). We may assume that \( \bar{a}_0 = \bar{a} \). \( \square \)
3. Superstability

We would like to find a notion of superstability, which would imply all the usual properties of non-forking for ↓, especially local character. We will suggest notions of superstability and weak superstability, and discuss the relation between these notions. We will also assume simplicity and weak stability in at least one cardinal, as we did in the previous section. Note that our notion of superstability uses the fact that LS(K) = ℵ₀. We also need simplicity to show that ℵ₀-stability implies superstability (see Corollary 3.28).

**Definition 3.1** (Superstability). We say that the class (K, ≼) is superstabled if it is weakly stable and the following holds.

Let (Aₙ)ₙ<ω be an increasing sequence of finite sets such that \( \bigcup_{n<\omega} Aₙ \) is a model, and let \( \bar{a} \) be a tuple. Then there is \( n < \omega \) such that \( \bar{a} \downarrow_{Aₙ} A_{n+1} \).

**Definition 3.2** (Weak superstability). We say that the class (K, ≼) is weakly superstabled if there is a cardinal \( \lambda \) such that (K, ≼) is weakly stable in all cardinals above \( \lambda \).

**Lemma 3.3** (Local character for models). Assume that a simple finitary (K, ≼) is superstable.

Let \( A \) be a model and \( \bar{a} \) a tuple. There is finite \( A' \subset A \) such that \( \bar{a} \downarrow_A A' \).

**Proof.** Let \( \bar{a} \) and \( A \) witness the contrary.

We recall the so called presentation theorem for abstract elementary classes with LS(K) = ℵ₀. This is the main tool in general abstract elementary classes introduced by Shelah. There is a vocabulary \( \tau^* \) with \( n \)-ary function symbols \( F^k_n \) for \( k, n < \omega \) such that for each model \( B \) in K there is a \( \tau^* \)-structure \( B^* \) such that \( B^* \upharpoonright \tau = B \) and whenever a subset \( B \subset B^* \) is closed under the functions \( (F^k_n)^{B^*} \), then \( B \preceq_K B \). Let \( \mathcal{A}^* \in K^* \) be such that \( \mathcal{A}^* \upharpoonright \tau = \mathcal{A} \). Define increasing and finite sets \( A_n \subset \mathcal{A} \), \( n < \omega \), such that

1. \( (F^k_n)^{\mathcal{A}^*}([A_n]^m) \subset A_{n+1} \) for \( k, m \leq n \)
2. \( \bar{a} \not\upharpoonright_{A_n} A_{n+1} \) for all \( n < \omega \).

We can take \( A_0 = \emptyset \). Assume we have defined \( A_k \) for \( k \leq n \). By assumption, \( \bar{a} \not\upharpoonright_{A_n} \mathcal{A} \), and by Proposition 2.23 there is finite \( A'_{n+1} \subset \mathcal{A} \) such that \( \bar{a} \not\upharpoonright_{A_n} A'_{n+1} \).

We take

\[
A_{n+1} = A'_{n+1} \cup \{(F^k_m)^{\mathcal{A}^*}([A_n]^m) : k, m \leq n \}.
\]

Then \( \bar{a} \not\upharpoonright_{A_n} A_{n+1} \) holds by monotonicity.

Finally \( \bigcup_{n<\omega} A_n \) is closed under the functions \( (F^k_m)^{\mathcal{A}^*} \), for \( m, k < \omega \), and thus is a model. We get a contradiction with superstability. \( \square \)
**Corollary 3.4.** Assume that a simple finitary \((\mathbb{K}, \preceq_{\mathbb{K}})\) is superstable. Then it is also weakly superstable.

Furthermore, let \(L(\mathbb{K})\) be a cardinal such that there are at most \(L(\mathbb{K})\) many Lascar strong types over any finite set. For any \(\kappa \geq L(\mathbb{K})\), there are at most \(\kappa\) many weak Lascar strong types over a set of size \(\kappa\).

**Proof.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is superstable. Let \(L(\mathbb{K})\) be the cardinal as above. (By the argument in the beginning of section 3.4 we know that \(L(\mathbb{K}) < H_{\kappa}\).) We show the latter claim and thus \((\mathbb{K}, \preceq_{\mathbb{K}})\) is weakly stable in each \(\kappa \geq L(\mathbb{K})\).

Let \((\bar{a}_i)_{i \leq \kappa^+}\) be finite tuples and \(\mathcal{A}\) a model such that \(|\mathcal{A}| = \kappa \geq L(\mathbb{K})\). It is enough to find \(i < j < \kappa^+\) such that \(Lst^w(\bar{a}_i/\mathcal{A}) = Lst^w(\bar{a}_j/\mathcal{A})\). By Local character for models, there are finite \(E_i, E_j\) such that

\[
\bar{a}_i \downarrow_{E_i} \mathcal{A} \quad \text{for each } i < \kappa^+.
\]

Since there are only \(\kappa\)-many finite subsets of \(\mathcal{A}\), we can find a subsequence \((\bar{a}_{i_k})_{k < \kappa^+}\) such that \(E_{i_k} = E\) for some fixed finite \(E \subset \mathcal{A}\) for all \(k < \kappa^+\). There are only \(L(\mathbb{K})\)-many different Lascar strong types over \(E\), and thus there are \(k, k' < \kappa^+\) such that \(Lst(\bar{a}_{i_k}/E) = Lst(\bar{a}_{i_{k'}}/E)\). But now by stationarity of Lascar strong types, \(Lst^w(\bar{a}_{i_k}/\mathcal{A}) = Lst^w(\bar{a}_{i_{k'}}/\mathcal{A})\). \(\square\)

We collect here the properties of \(\downarrow\) that we gain from simplicity and superstability. Since we only have local character for models, these properties are still incomplete: we only have the independence calculus for models and finite sets. We will gain local character and the full independence calculus for all sets in Theorem 3.13, where we assume also the Tarski-Vaught property.

**Theorem 3.5.** Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a simple, superstable, finitary AEC. Then the relation \(\downarrow\) has the following properties.

1. **Invariance:** If \(A \triangleleft_C B\), then \(f(A) \downarrow_{f(C)} f(B)\) for an \(f \in \text{Aut}(\mathfrak{M})\).
2. **Monotonicity:** If \(A \triangleleft_B D\) and \(B \subset C \subset D\) then \(A \triangleleft_C D\) and \(A \triangleleft_B C\).
3. **Local character for models:** For any finite \(\bar{a}\) and any \(B\) there exists a finite \(E \subset B\) such that \(\bar{a} \downarrow_{E} B\).
4. **Transitivity:** Let \(B \subset C \subset D\). If \(A \triangleleft_B C\) and \(A \triangleleft_C D\), then \(A \triangleleft_B D\).
5. **Restricted finite character:** Assume that \(C\) is finite. \(A \triangleleft_C B\) if and only if \(\bar{a} \downarrow_C \bar{b}\) for every finite \(\bar{a} \in A\) and \(\bar{b} \in B\).
6. **Finite character for models:** Assume that \(\mathcal{C}\) is a model. \(A \triangleleft_{\mathcal{C}} B\) if and only if \(\bar{a} \downarrow_{\mathcal{C}} \bar{b}\) for every finite \(\bar{a} \in A\) and \(\bar{b} \in B\).
7. **Reflexivity for finite sets:** Assume that \(C\) is finite and \(tp^w(\bar{a}/C)\) is not bounded. Then \(\bar{a} \not\triangleleft_C \bar{a}\).
(8) **Reflexivity for models:** Assume that \( \mathcal{C} \) is a model and \( \text{tp}^w(a/\mathcal{C}) \) is not bounded. Then \( \bar{a} \not\models_{\mathcal{C}} \bar{a} \).

(9) **Stationarity:** If \( \text{Lstp}^w(a/C) = \text{Lstp}^w(b/C) \), \( \bar{a} \downarrow_C B \) and \( \bar{b} \downarrow_C B \), then \( \text{Lstp}^w(a/B \cup C) = \text{Lstp}^w(b/B \cup C) \).

(10) **Extension for finite sets:** For any finite \( C \), \( \bar{a} \) and any \( B \) containing \( C \), there is \( \bar{b} \) such that \( \text{Lstp}(\bar{b}/C) = \text{Lstp}^w(\bar{a}/C) \) and \( \bar{b} \downarrow_C B \).

(11) **Extension for models:** For any \( \bar{a} \), \( \mathcal{C} \) a model and \( B \) containing \( \mathcal{C} \) there is \( \bar{b} \) such that \( \text{Lstp}^w(\bar{b}/\mathcal{C}) = \text{Lstp}^w(\bar{a}/\mathcal{C}) \) and \( \bar{b} \downarrow_{\mathcal{C}} B \).

(12) **Restricted symmetry:** Assume that \( \mathcal{C} \) is finite. \( A \downarrow_{\mathcal{C}} B \) if and only if \( B \downarrow_{\mathcal{C}} A \).

(13) **Symmetry over models:** Assume that \( \mathcal{C} \) is a model. \( A \downarrow_{\mathcal{C}} B \) if and only if \( B \downarrow_{\mathcal{C}} A \).

**Proof.** Items (1) and (2) were studied in Proposition 2.6, and local character for models was proved in Lemma 3.3. Transitivity was stated in Proposition 2.26 and restricted finite character in Proposition 2.23. The other direction of finite character for models follows from monotonicity. Assume that \( \mathcal{C} \) is a model and that \( A \not\subseteq_{\mathcal{C}} B \). By definition, there is \( \bar{a} \in A \) such that \( \bar{a} \not\subseteq_{\mathcal{C}} \bar{b} \). By local character for models, we can choose finite \( E \subset \mathcal{C} \) such that \( \bar{a} \downarrow_E \mathcal{C} \). Let \( \bar{b} \in \mathcal{C} \cup B \) be finite. We have that \( E \subset \mathcal{C} \subset \mathcal{C} \cup \{ \bar{b} \} \), \( \bar{a} \downarrow_E \mathcal{C} \) and \( \bar{a} \not\subseteq_{\mathcal{C}} \bar{b} \). Hence \( \bar{a} \downarrow_E \bar{b} \) by transitivity. Since \( \bar{b} \) was arbitrary, we have \( \bar{a} \downarrow_E \mathcal{C} \cup B \) by restricted finite character, a contradiction. This proves finite character for models.

Reflexivity for finite sets is Lemma 2.9(b). For models, assume the contrary. Let \( \text{tp}^w(a/\mathcal{C}) \) be unbounded such that \( \mathcal{C} \) is a model and \( \bar{a} \not\subseteq_a \bar{a} \). By local character for models there is finite \( E \subset \mathcal{C} \) such that \( \bar{a} \downarrow_E \mathcal{C} \). By transitivity and monotonicity, \( \bar{a} \downarrow_E \bar{a} \). Since the type \( \text{tp}^w(a/E) \) cannot be bounded, we get a contradiction with 7.

Stationarity follows from Proposition 2.30. Extension for finite sets is Corollary 2.12 with simplicity. We prove extension over models. By local character for models, there is finite \( E \subset \mathcal{C} \) such that \( \bar{a} \downarrow_E \mathcal{C} \). Then by extension for finite sets there is \( \bar{b} \) realizing \( \text{Lstp}(\bar{a}/E) \) such that \( \bar{b} \downarrow_E D \). But now by stationarity, \( \bar{b} \) realizes also \( \text{Lstp}^w(\bar{a}/\mathcal{C}) \). Finite symmetry follows from Proposition 2.20 and restricted finite character. To prove symmetry over models, it is enough to prove that \( \bar{a} \not\subseteq_{\mathcal{C}} \bar{b} \) implies \( \bar{b} \not\subseteq_{\mathcal{C}} \bar{a} \) for each \( \bar{a} \in A \) and \( \bar{b} \in B \). Then symmetry over models follows by finite character for models. Assume that \( \bar{a} \not\subseteq_{\mathcal{C}} \bar{b} \). By local character for models, there is finite \( E \subset \mathcal{C} \) such that \( \bar{a} \downarrow_E \mathcal{C} \) and \( \bar{b} \downarrow_E \mathcal{C} \). Then also \( \bar{a} \downarrow_E \mathcal{C} \cup \bar{b} \) by transitivity. Let \( \bar{c} \in \mathcal{C} \) be an arbitrary finite tuple. We get \( \bar{c} \downarrow_E \bar{b} \) by symmetry and since \( \bar{a} \downarrow_E \{ \bar{c}, \bar{b} \} \), Pairs Lemma implies that \( \bar{c} \bar{a} \downarrow_E \bar{b} \). Hence by symmetry again, we have that \( \bar{b} \downarrow_E \{ \bar{a}, \bar{c} \} \) for each finite \( \bar{c} \in \mathcal{C} \). This implies \( \bar{b} \downarrow_E \mathcal{C} \cup \{ \bar{a} \} \) by restricted finite character, and thus \( \bar{b} \not\subseteq_{\mathcal{C}} \bar{a} \). \[\square\]
We recall the following lemma from [8]. The proof uses finite character of \((\mathbb{K}, \preceq_\mathbb{K})\), and this is the first place in this paper where we really use it. Without finite character we should assume that each \(A_n\) is a model. This lemma is needed in several places where we deal with models built out of finite sets. Especially this lemma is needed in the tree constructions in Lemma 3.7 and Proposition 3.26, and in the essential Proposition 3.11, which we use to build primary models.

**Lemma 3.6.** Assume that \((\mathbb{K}, \preceq_\mathbb{K})\) is a finitary AEC. Let \((A_n : n < \omega)\) be an increasing sequence of sets such that \(\bigcup_{n<\omega} A_n\) is a model in \(\mathbb{K}\). Let \((\bar{b}_n)_{n<\omega}\) be a sequence of finite tuples of the same length, such that
\[
\text{tp}^\mathbb{K}(\bar{b}_m/A_n) = \text{tp}^\mathbb{K}(\bar{b}_n/A_n), \quad \text{for each } n < m < \omega.
\]
Then there exists a tuple \(\bar{a}\) such that
\[
\text{tp}^\mathbb{K}(\bar{a}/A_n) = \text{tp}^\mathbb{K}(\bar{b}_n/A_n), \quad \text{for each } n < \omega.
\]

We give a sufficient condition for \((\mathbb{K}, \preceq_\mathbb{K})\) being superstable. We will also see in Theorem 3.38, that this condition is implied by a very weak version of categoricity. Finite character is needed here, since we work with finite sets, not models.

**Lemma 3.7.** Let \((\mathbb{K}, \preceq_\mathbb{K})\) be a simple finitary AEC. Assume that there are infinite cardinals \(\kappa\) and \(\lambda\) such that \(\kappa^{\aleph_0} \leq \lambda, \lambda^{\aleph_0} > \lambda\), \((\mathbb{K}, \preceq_\mathbb{K})\) is weakly stable in \(\lambda\) and the following holds for \(\kappa\).

For all \(\bar{a}\) and finite \(A\) there is a strongly indiscernible sequence \((\bar{a}_i)_{i<\kappa}\) such that for any \(\bar{b}\) the set \(\{i < \kappa : \bar{b} \not\equiv_A \bar{a}_i\}\) has size strictly smaller than \(\kappa\).

Then \((\mathbb{K}, \preceq_\mathbb{K})\) is superstable.

**Proof.** Assume to the contrary, that \(A_n\) are increasing and finite, \(\bigcup_{n<\omega} A_n\) is a model and \(\bar{a} \not\equiv_A A_{n+1}\) for each \(n < \omega\).

We define sets \(A^k_{\eta n}\) and tuples \(\bar{a}_{\eta n}\) for all \(\eta : \omega \rightarrow \lambda \) and \(n \leq k < \omega\) such that
\begin{enumerate}
  \item \(A^k_{\eta n} \subset A^{k+1}_{\eta n}\) are finite and the type \(\text{tp}^\mathbb{K}(A^{k+1}_{\eta n}/A^k_{\eta n})\) is unbounded,
  \item \(\bar{a}_{\eta 0} = \bar{a}\) and \(A^k_{\eta 0} = A_k\) for all \(k < \omega\),
  \item \(\text{tp}^\mathbb{K}(\bar{a}_{\eta n+1}/A^n_{\eta n}) = \text{tp}^\mathbb{K}(\bar{a}_{\eta n}/A^n_{\eta n})\) and \(\bar{a}_{\eta n+1} \not\equiv_A A^k_{\eta n+1}\),
  \item \(A^n_{\eta n+1} = A^n_{\eta n}\) and the sequence \((A^{n+1}_{\eta n+1})_{\eta(n)<\lambda}\) is strongly \(A^n_{\eta n}\)-indiscernible,
  \item for all \(\bar{b}\), we have that \(|\{\eta(n) < \lambda : \bar{b} \not\equiv_A A_{\eta[n]}^{n+1}\}| < \kappa\).
\end{enumerate}

First define \(\bar{a}_{\eta 0}\) and \(A^k_{\eta 0}\) as in (2). Assume we have defined \(\bar{a}_{\eta m}\) and \(A^k_{\eta m}\) for all \(\eta : \omega \rightarrow \lambda\) and \(m \leq k < \omega\) for \(m \leq n\). Let \((\bar{b}_i)_{i<\kappa}\) be the \(A^n_{\eta n}\)-indiscernible sequence implied by the assumption, such that \(\bar{b}_0 = A^n_{\eta n}\). We can stretch this sequence to \((\bar{b}_i)_{i<\lambda'}\), and still, for any \(\bar{b}\), the set
\[
\{i < \lambda' : \bar{b} \not\equiv_A A_{\eta[n]}^{n+1} \bar{b}_i\} \]

is of size strictly less than \( \kappa \). There is an automorphism \( f^n_i \in \text{Aut}( \mathcal{M}/A^n_{\eta|n} ) \) mapping \( A^{n+1}_{\eta|n} \) to \( \tilde{b}_i \) for each \( i < \lambda \), and we can take \( f^0_0 = \text{Id}_{\mathcal{M}} \). When \( \eta(n) = i \), we define
\[
A^n_{\eta|n+1} = A^n_{\eta|n},
\]
\[
A^k_{\eta|n+1} = f^n_i(A^k_{\eta|n}) \text{ for each } n < k < \omega; \text{ and}
\]
\[
\bar{a}_{\eta|n+1} = f^n_i(\bar{a}_{\eta|n}).
\]
Now we are done with the construction.

For \( \eta: \omega \to \lambda \) and \( n < \omega \), denote \( g^n_\eta = f^n_{\eta(n)} \circ \ldots \circ f^1_{\eta(1)} \circ f^0_{\eta(0)} \in \text{Aut}(\mathcal{M}) \), where the automorphisms \( f^n_i \) are as in the previous construction. Always \( g^{n+1} \upharpoonright A_n = g^n \upharpoonright A_n \). By finite character of \( (\mathbb{K}, \prec \mathbb{K}) \), the set
\[
\bigcup_{n<\omega} A^n_{\eta|n} = \bigcup_{n<\omega} g^n_\eta(A_n)
\]
is a model. Using Lemma 3.6 and (3), we find \( \bar{a}_\eta \) for each \( \eta: \omega \to \lambda \) such that
\[
\text{tp}^g(\bar{a}_\eta/A^n_{\eta|n}) = \text{tp}^g(\bar{a}_{\eta|n}/A^n_{\eta|n}) \text{ for each } n < \omega.
\]
We look at the types of the tuples \( \bar{a}_\eta \) over the set
\[
B = \bigcup_{\eta: \omega \to \lambda, n, k < \omega} A^k_{\eta|n},
\]
which has size \( \lambda \). First we claim that for a fixed \( \eta: \omega \to \lambda \), there are less than \( \kappa^{\aleph_0} \) many \( \bar{a}_{\eta'} \) realizing \( \text{tp}^w(\bar{a}_\eta/B) \).

We prove the claim by pruning the tree of \( \eta' \)'s at one level \( n < \omega \) at the time leaving out all the branches \( \eta' \) such that \( \bar{a}_{\eta'} \) cannot realize \( \text{tp}^w(\bar{a}_\eta/B) \) for a simple reason. We leave at most \( \kappa^n \) branches at each level \( n \), and the final tree will be of size at most \( \kappa^{\aleph_0} \). At level 0 there is only one branch \( \eta \upharpoonright 0 \). Assume that at level \( n < \omega \) there are left at most \( \kappa^n \) branches \( \eta' \upharpoonright n: n \to \lambda \). Let \( \eta' \upharpoonright n \) be one such branch with possible extensions \( \eta'(n) < \lambda \). If \( \bar{a}_\eta \) realizes \( \text{tp}^w(\bar{a}_{\eta'|n+1}/A^n_{\eta'|n+1}) \), we must have that \( \bar{a}_\eta \not\in A^n_{\eta'|n+1} A^{n+1}_{\eta'|n+1} \) by (3). But by (5), this can only happen for at most \( \kappa \) many \( \eta'(n) < \lambda \). We leave only those extensions to the tree. We do the pruning for each branch \( \eta' \upharpoonright n \) of the tree and are left with at most \( \kappa^{n+1} \) many branches \( \eta' \upharpoonright n+1 \) at level \( n+1 \). This proves the claim.

Let us partition the tuples \( \bar{a}_\eta \) into equivalence classes according to their weak types over the set \( B \). Now by \( \lambda \)-stability, there are at most \( \lambda \)-many classes, and by the previous claim, each class is of size at most \( \kappa^{\aleph_0} \). This is a contradiction, since the number of tuples is \( \lambda^{\aleph_0} > \lambda \times \kappa^{\aleph_0} \). \( \square \)
3.1. Tarski-Vaught -property. In [7] we used finite character and $\aleph_0$-stability to construct models. These properties imply that whenever a set $A$ has the property that each Galois type over each finite subset of $A$ is satisfied in $A$, then $A$ is actually a model. Since in this case there are only countably many Galois types over each finite set, we got a useful tool for extending an arbitrary set $A$ to a model of size $|A| + \aleph_0$. Here we need a similar property. The finite character property generalizes the idea that $\preceq_K$ would be induced by a language with finitely many free variables in each formula. Respectively the Tarski-Vaught -property can be seen as a generalization of the 'countable' Tarski-Vaught criterion for elementary classes: To check whether a set is an elementary submodel, it is enough to see that it is existentially closed respect to all formulas in the countable language. We use sets of Galois types over the empty set to generalize the notion of a formula in a language.

**Definition 3.8 (Formula).** By an $n$-formula $\phi$ we mean a set of Galois types $tp^0(\bar{a}/\emptyset)$ of $n$-tuples $\bar{a}$ over the empty set. For an $n$-tuple $\bar{b} \in \mathcal{B}$, denote $\mathcal{B} \models \phi(\bar{b})$, if $tp^0(\bar{b}/\emptyset, \mathcal{B}) \in \phi$.

**Assumption 3.9 (Tarski-Vaught -property).** Let $S$ be a set of formulas. We say that a set $A \subset \mathfrak{M}$ is $S$-saturated, if the following holds.

For any finite $\bar{a} \in A$, $\bar{b} \in \mathfrak{M}$ and $\phi \in S$, if $\mathfrak{M} \models \phi(\bar{a} \bar{b})$, there is $\bar{d} \in A$ such that $\mathfrak{M} \models \phi(\bar{a} \bar{d})$.

We call as the Tarski-Vaught -property the following: There is a countable set $S$ of formulas such that any $S$-saturated subset $A \subset \mathfrak{M}$ is a $K$-elementary submodel of $\mathfrak{M}$.

**Remark 3.10.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is an $\aleph_0$-stable finitary AEC. Then it has the Tarski-Vaught -property.

**Proof.** We can take as $S$ the set of all singletons of Galois types over the empty set. By $\aleph_0$-stability, there are only countable many of them. By finite character, any $\aleph_0$-saturated subset is a model (Lemma 3.8 of [8]). \qed

The following useful Proposition uses finite character of $(\mathbb{K}, \preceq_{\mathbb{K}})$ in the form of Lemma 3.6.

**Proposition 3.11.** Assume that finitary $(\mathbb{K}, \preceq_{\mathbb{K}})$ is simple, superstable and has the Tarski-Vaught -property. Let $(A_i)_{i<\omega}$ be finite and increasing and let $(\bar{a}_i)_{i<\omega}$ be tuples such that for $i < j$, $\text{Lstp}(\bar{a}_j/A_i) = \text{Lstp}(\bar{a}_i/A_i)$. Then there is some $i < \omega$ such that $\bar{a}_{i+1} \downarrow_{A_i} A_{i+1}$.

**Proof.** We define tuples $\bar{b}_i$ and finite increasing sets $B_i$ as follows

1. $B_i = A_i \cup \bigcup_{j \leq i} \bar{b}_j$ for each $i < \omega$. 


(2) \( \bar{b}_i \downarrow B_i \left( \bigcup_{j<\omega} \bar{a}_j \cup \bigcup_{j<\omega} A_j \right) \) for each \( i < \omega \).

(3) \( B = \bigcup_{i<\omega} B_i \) satisfies the following: For each finite \( \bar{b} \in B, \bar{d} \in \mathfrak{M} \) and \( \phi \in S \) such that \( \mathfrak{M} \models \phi(\bar{b}d) \) there is \( \bar{c} \in B \) such that \( \mathfrak{M} \models \phi(\bar{b}\bar{c}) \).

We have defined \( B_n \), let \( (\bar{c}_j^n)_{j<\omega} \) be such tuples that whenever there exists a tuple \( \bar{c} \) such that \( \mathfrak{M} \models \phi(\bar{b}, \bar{c}) \) for some \( \phi \in S \) and finite \( \bar{b} \in B_n \), then one such \( \bar{c} \) is listed as \( \bar{c}_j^n \) for some \( j < \omega \). Then let \( \bar{d}_n \) contain \( \bar{c}_j^n \) for every \( j, n' \leq n \). If each \( \text{tp}^g(\bar{d}_n/B_n) \) is realized in \( B \), then clearly (3) holds.

First let \( B_0 = A_0 \). Assume we have defined \( B_n \) and \( \bar{b}_i \) for each \( i < n \). Since \( \bar{d}_n \downarrow B_n \) by simplicity, we can use extension to get \( \bar{b}_n \) such that \( \text{tp}^g(\bar{b}_n/B_n) = \text{tp}^g(\bar{d}_n/B_n) \) and (2) holds for \( n \). Then let \( B_{n+1} = A_n \cup \bigcup_{i<n} \bar{b}_i \). We are done with the construction.

We claim that when \( i, j \geq n \),

\[ \bar{a}_i \downarrow A_j B_n. \]

We prove the claim by induction on \( n \) and for all \( i, j \geq n \) simultaneously. By simplicity, \( \bar{a}_i \downarrow A_j A_j \) for each \( i, j \geq n \). Since \( B_0 \subset A_j \) for each \( j \), this gives the claim for \( n = 0 \). Assume we have shown the claim for \( n \) and let \( i, j \geq n+1 \). By (2), \( \bar{b}_n \downarrow B_n \bar{a}_i \cup A_j \bar{b}_n \). By (2), \( \bar{b}_n \downarrow B_n \bar{a}_i \cup A_j \bar{b}_n \). Induction and finite transitivity give that \( \bar{a}_i \downarrow A_j B_n \bar{b}_n \). Since \( B_{n+1} = B_n \bar{b}_n A_{n+1} \subset B_n \bar{b}_n A_j \), this gives the claim.

Now we have that \( \mathfrak{Lst}(\bar{a}_j/A_n) = \mathfrak{Lst}(\bar{a}_j/A_n), \bar{a}_n \downarrow A_n B_n \) and \( \bar{a}_j \downarrow A_n B_n \) for each \( n < j < \omega \), and thus get by stationarity, that

\[ \text{tp}^g(\bar{a}_j/B_n) = \text{tp}^g(\bar{a}_n/B_n) \] for each \( n < j < \omega \).

Since \( B = \bigcup_{n<\omega} B_n \) is a model by (3), we can use Lemma 3.6 to get a tuple \( \bar{a} \) such that

\[ \text{tp}^g(\bar{a}/B_n) = \text{tp}^g(\bar{a}_n/B_n) \] for each \( n < \omega \).

Since \( B \) is a model, by superstability there is \( n < \omega \) such that \( \bar{a} \downarrow B_n B_{n+1} \), and furthermore by invariance, \( \bar{a}_{n+1} \downarrow B_n B_{n+1} \). By the previous claim, \( \bar{a}_{n+1} \downarrow A_n B_n \), and thus by finite transitivity, \( \bar{a}_{n+1} \downarrow A_n B_{n+1} \). Since \( A_{n+1} \subset B_{n+1} \), this \( \bar{a}_{n+1} \) is the one required for the proposition.

We can easily derive the following corollary, using the restricted finite character property of \( \downarrow \).

**Corollary 3.12** (Local character). Let \( (\mathfrak{K}, \preceq_{\mathfrak{K}}) \) be simple, superstable, finitary AEC with the Tarski-Vaught-property. Assume that \( \bar{a} \) is a tuple and \( A \) is an arbitrary set. Then there is finite \( E \subset A \) such that \( \bar{a} \downarrow E A \).
Finally we get the usual properties of non-forking for complete types over arbitrary sets. The proof of the following is analogous to the proof of Theorem 3.5.

**Theorem 3.13.** Assume that \((\mathcal{K}, \preceq_{\mathcal{K}})\) is a simple, superstable, finitary AEC with the Tarski-Vaught -property. Then the relation \(\downarrow\) has the following properties.

1. **Invariance:** If \(A \downarrow_{C} B\), then \(f(A) \downarrow_{f(C)} f(B)\) for an \(f \in \text{Aut}(\mathfrak{M})\).
2. **Monotonicity:** If \(A \downarrow_{B} D\) and \(B \subset C \subset D\) then \(A \downarrow_{C} D\) and \(A \downarrow_{B} C\).
3. **Local character:** For any finite \(\bar{a}\) and any \(B\) there exists a finite \(E \subset B\) such that \(\bar{a} \downarrow_{E} B\).
4. **Transitivity:** Let \(B \subset C \subset D\). If \(A \downarrow_{B} C\) and \(A \downarrow_{C} D\), then \(A \downarrow_{B} D\).
5. **Finite character:** \(A \downarrow_{C} B\) if and only if \(\bar{a} \downarrow_{C} \bar{b}\) for every finite \(\bar{a} \in A\) and \(\bar{b} \in B\).
6. **Reflexivity:** If \(\text{tp}^{w}(\bar{a}/A)\) is not bounded, then \(\bar{a} \not\downarrow_{A} \bar{a}\).
7. **Stationarity:** If \(\text{Lstp}^{w}(\bar{a}/C) = \text{Lstp}^{w}(\bar{b}/C), \bar{a} \downarrow_{C} B\) and \(\bar{b} \downarrow_{C} B\), then \(\text{Lstp}^{w}(\bar{a}/B \cup C) = \text{Lstp}^{w}(\bar{a}/B \cup C)\).
8. **Extension:** For any \(\bar{a}, C\) and \(B\) containing \(C\) there is \(\bar{b}\) such that \(\text{Lstp}^{w}(\bar{b}/C) = \text{Lstp}^{w}(\bar{a}/C)\) and \(\bar{b} \downarrow_{C} B\).
9. **Symmetry:** \(A \downarrow_{C} B\) if and only if \(B \downarrow_{C} A\).

### 3.2. Weak Lascar strong type and superstability

In this chapter we study the behaviour of weak Lascar strong types in superstable simple finitary AEC. First we study when so called abstract weak Lascar strong types are realized. We say that \(p\) is an abstract weak Lascar strong type over \(A\), if \(p\) is a collection

\[
p = \{\text{Lstp}(\bar{a}/B) : B \subset A\text{ finite}\},
\]

where \(B \subset B' \subset A\) implies \(\text{Lstp}(\bar{a}/B) = \text{Lstp}(\bar{a}/B')\). For finite \(B \subset A\), the type \(\text{Lstp}(\bar{a}/B) \in p\) can be also denoted as \(p \upharpoonright B\). We say that \(p\) is realized by \(\bar{a}\), if \(\text{Lstp}(\bar{a}/B) \in p\) for all finite \(B \subset A\). We will show that abstract weak Lascar strong types over models are realized in superstable simple finitary classes, and if the class in addition has the Tarski-Vaught -property, all abstract weak Lascar strong types are realized. For these proofs we need versions of local character for abstract types. When \(p\) is an abstract weak Lascar strong type over \(A\), we say that \(p\) is independent of \(A\) over \(E\), written

\[
p \downarrow_{E} A,
\]

if \(a_{B} \downarrow_{E} B\) for all finite \(B \subset A\) such that \(E \subset B\) and \(\text{Lstp}(\bar{a}/B) \in p\).

**Lemma 3.14.** Assume that \((\mathcal{K}, \preceq_{\mathcal{K}})\) is a simple, superstable finitary AEC.

Let \(\mathcal{A}\) be a model and \(p\) an abstract weak Lascar strong type over \(A\). There is finite \(E \subset \mathcal{A}\) such that \(p \downarrow_{E} \mathcal{A}\).
Proof. Let and $\mathcal{A}$ and $p$ witness the contrary. The proof for this Lemma is analogous to the proof of Lemma 3.3. Let again $F^k_n$ for $k, n < \omega$ be function symbols from the presentation theorem and $\mathcal{A}^*$ be the extension of $\mathcal{A}$. Define increasing and finite sets $A_n \subset \mathcal{A}$ and tuples $\bar{a}_n$ for $n < \omega$ such that

1. $\text{Lstp}(\bar{a}_n/A_n) \in p$ for each $n < \omega$,
2. $(F^k_m)^{\mathcal{A}^*}(A_n) \subset A_{n+1}$ for $k, m \leq n$ and
3. $\bar{a}_{n+1} \not \models_{A_n} A_{n+1}$ for each $n < \omega$.

We can take $A_0 = \emptyset$ and $\bar{a}_0$ realizing $p \upharpoonright \emptyset$. Assume we have defined $A_k$ for $k \leq n$. By assumption, $p \not \models_{A_n} \mathcal{A}$, and thus there is some finite $A'_{n+1} \subset \mathcal{A}$ such that $A_n \subset A'_{n+1}$ and $(p \upharpoonright A'_{n+1}) \not \models_{A_n} A_{n+1}$. Let

$$A_{n+1} = A'_{n+1} \cup \{(F^k_m)^{\mathcal{A}^*}(A_n) : k, m \leq n\}$$

and $\bar{a}_{n+1}$ be a tuple realizing $p \upharpoonright A_{n+1}$. Since $\bar{a}_{n+1}$ realizes also $p \upharpoonright A'_{n+1}$, we have that $\bar{a}_{n+1} \not \models_{A_n} A_{n+1}$, and thus (3) holds by monotonicity.

Finally $\bigcup_{n<\omega} A_n$ is closed under the functions $(F^k_m)^{\mathcal{A}^*}$, for $m, k < \omega$, and thus is a model. We can use Lemma 3.6 to find $\bar{a}$ realizing $\text{tp}^g(\bar{a}_n/A_n)$ for each $n < \omega$. Now $\bar{a}$ and $(A_n)_{n<\omega}$ contradict superstability. \qed

With the Tarski-Vaught-property we can prove a stronger lemma. The proof is analogous to the previous one. We do not need the functions $F^k_m$ to contradict superstability, but make a contradiction with Proposition 3.11 instead.

**Lemma 3.15.** Assume that $(\mathcal{K}, \preceq_\mathcal{K})$ is a simple, superstable finitary AEC with the Tarski-Vaught-property.

Let $p$ be an abstract weak Lascar strong type over a set $A$. There is finite $E \subset A$ such that $p \downarrow_E A$.

Now we can use extension to prove the following Theorem.

**Theorem 3.16.** Assume that $(\mathcal{K}, \preceq_\mathcal{K})$ is a superstable, simple, finitary AEC. Then each abstract weak Lascar strong type over a model is realized. If $(\mathcal{K}, \preceq_\mathcal{K})$ in addition has the Tarski-Vaught-property, all abstract weak Lascar strong types are realized.

Proof. We prove the first claim. Then it is clear how to prove the second claim using Lemma 3.15. Let $p$ be an abstract weak Lascar strong type over a model $\mathcal{A}$. By Lemma 3.14, there is finite $E \subset \mathcal{A}$ such that $p \downarrow_E \mathcal{A}$. Let $\bar{b}$ realize $p \upharpoonright E$. By simplicity, $\bar{b} \downarrow_E E$, and thus by Corollary 2.12, there is $\bar{a}$ realizing $\text{Lstp}(\bar{b}/E)$ such that $\bar{a} \downarrow_E \mathcal{A}$. This $\bar{a}$ realizes $p$ by stationarity. \qed
Another consequence of superstability and the Tarski-Vaught-property is that weak Lascar strong type is a stronger notion than Galois type over all countable sets. The proof of this theorem is a similar construction as in the \(\aleph_0\)-stable case, when we proved that equivalent Galois types imply equivalent weak types over countable models, see [8]. Again we introduce a notion of an isolated type.

**Definition 3.17** (Isolation over a pair). We say that the type \(\text{Lstp}^w(\bar{a}\bar{c}/A)\) is isolated over the pair \((\bar{c},E)\), for finite \(E \subset A\), if for every \(\bar{b}\) such that \(\text{Lstp}(\bar{b}\bar{c}/E) = \text{Lstp}(\bar{a}\bar{c}/E)\) we have that \(\text{Lstp}^w(\bar{b}\bar{c}/A) = \text{Lstp}^w(\bar{a}\bar{c}/A)\).

We remark that \(\text{Lstp}(\bar{b}\bar{c}/E) = \text{Lstp}(\bar{a}\bar{c}/E)\) does not necessarily imply that \(\text{Lstp}(\bar{b}/E \cup \bar{c}) = \text{Lstp}(\bar{a}/E \cup \bar{c})\), although the converse holds. Hence the previous notion of isolation is needed for the proof of Theorem 3.19.

**Proposition 3.18.** Let \((\mathcal{K}, \preceq_{\mathcal{K}})\) be superstable simple finitary AEC with the Tarski-Vaught-property.

For every set \(A\), finite \(B \subset A\) and tuples \(\bar{b}, \bar{c}\), there is \(\bar{a}\) and finite \(E \subset A\) such that \(\bar{a}\bar{c}\) realizes \(\text{Lstp}(\bar{b}\bar{c}/B)\) and \(\text{Lstp}^w(\bar{a}\bar{c}/A)\) is isolated over the pair \((\bar{c},A)\).

**Proof.** Let \(B, A, \bar{b}\) and \(\bar{c}\) witness the contrary. We define finite and increasing sets \(A_n \subset A\) and tuples \(\bar{a}_n\) for \(n < \omega\) such that

1. \(\bar{b} = \bar{a}_0\) and \(B \subset A_0\),
2. \(\text{Lstp}(\bar{a}_{n+1}\bar{c}/A_n) = \text{Lstp}(\bar{a}_n\bar{c}/A_n)\),
3. \(\bar{a}_{n+1}\bar{c} \downarrow_{A_n} A_{n+1}\) and
4. \(\bar{c} \downarrow_{A_0} A\).

This construction will contradict Proposition 3.11. We do the construction as follows. First, by Local character, there is finite \(E' \subset A\) such that \(\bar{c} \downarrow_{E'} A\). We take \(\bar{a}_0 = \bar{b}\) and \(A_0 = E' \cup B\). Assume we have defined \(A_m\) and \(\bar{a}_m\) for \(m \leq n\).

By Theorem 3.13(7) there is \(\bar{d}\) realizing \(\text{Lstp}(\bar{a}_n/A_n \cup \bar{c})\) such that \(\bar{d} \downarrow_{A_n \cup A} A\). We get that \(\bar{d}\bar{c}\) realizes \(\text{Lstp}(\bar{a}_n\bar{c}/A_n)\). Since \(\text{Lstp}^w(\bar{d}\bar{c}/A)\) cannot be isolated over the pair \((\bar{c},A_n)\), there is \(\bar{a}_{n+1}\) such that \(\text{Lstp}(\bar{a}_{n+1}\bar{c}/A_n) = \text{Lstp}(\bar{d}\bar{c}/A_n) = \text{Lstp}(\bar{a}_n\bar{c}/A_n)\) but \(\text{Lstp}^w(\bar{a}_{n+1}\bar{c}/A) \neq \text{Lstp}^w(\bar{d}\bar{c}/A)\).

Now we can’t have that \(\bar{a}_{n+1}\bar{c} \downarrow_{A_n} A\). Otherwise, since \(\bar{d} \downarrow_{A_n \cup A}\) and \(\bar{c} \downarrow A_n\), Pairs Lemma implies that \(\bar{d}\bar{c} \downarrow_{A_n} A\). But then \(\text{Lstp}^w(\bar{d}\bar{c}/A) = \text{Lstp}^w(\bar{a}_{n+1}\bar{c}/A)\) by stationarity, a contradiction. Thus by finite character there is finite \(A_{n+1} \subset A\) such that \(A_n \subset A_{n+1}\) \(\bar{a}_{n+1}\bar{c} \downarrow_{A_n} A_{n+1}\). We are done with the construction. \(\square\)

**Theorem 3.19.** Assume that \((\mathcal{K}, \preceq_{\mathcal{K}})\) is a simple, finitary superstable AEC with the Tarski-Vaught property. Let \(A\) be a countable set. Then \(\text{Lstp}^w(\bar{a}/A) = \text{Lstp}^w(\bar{b}/A)\) implies that \(\text{tp}^a(\bar{a}/A) = \text{tp}^a(\bar{b}/A)\).
Proof. Enumerate $A = \{c_n : n < \omega\}$. We define sequences $\bar{a}_n$ and finite $A_n \subset A$ for each $n < \omega$ such that

1. $\bar{a} = \bar{a}_0$, $\bar{a}_n$ is an initial segment of $\bar{a}_{n+1}$ and $c_n \in A_n \subset A_{n+1} \subset A$,
2. $\text{Lstp}^w(\bar{a}_{n+1}/A)$ is isolated over the pair $(\bar{a}_n, A)$ and
3. $\mathcal{B} = A \cup \bigcup_{n < \omega} \bar{a}_n$ is $S$-saturated for the countable set $S$ of formulas from the Tarski-Vaught-property:
   
   For all finite $\bar{b} \in \mathcal{B}$, $\bar{d} \in M$ and $\phi \in S$ such that $M \models \phi(\bar{b}\bar{d})$ there is $\bar{c} \in B$ such that $M \models \phi(\bar{b}\bar{c})$.

We define simultaneously tuples $(\bar{c}^n_j)_{j < \omega}$ and $\bar{d}_n \in M$.

First let $\bar{a}_0 = \bar{a}$ and $A_0 = \{c_0\}$. Assume we have defined $A_m, \bar{a}_m$ for $m \leq n$. Let $(\bar{c}^n_j)_{j < \omega}$ be such tuples that whenever there exists a tuple $\bar{c}$ such that $M \models \phi(\bar{b}, \bar{c})$ for some $\phi \in S$ and finite $\bar{b} \in B_n$, then one such $\bar{c}$ is listed as $\bar{c}^n_j$ for some $j < \omega$. Then let $\bar{d}_n$ be finite such that $\bar{c}^n_j \subset \bar{d}_n$ for all $k, j < n$. By Proposition 3.18 there is $\bar{a}'$ and finite $A' \subset A$ such that

$$\text{Lstp}(\bar{a}'\bar{a}_n/A_n) = \text{Lstp}(\bar{d}_n\bar{a}_n/A_n)$$

and $\text{Lstp}(\bar{a}'\bar{a}_n/A)$ is isolated over the pair $(\bar{a}_n, A')$. Let $A_{n+1} = A_n \cup A' \cup c_{n+1}$ and $\bar{a}_{n+1} = \bar{a}_n\bar{a}'$.

We are done with the first construction. Now clearly (1) and (2) hold. Also (3) holds by the construction and the fact that having the same Lascar strong type implies having the same Galois type. Thus

$$\mathcal{B} = A \cup \bigcup_{n < \omega} \bar{a}_n = \bigcup_{n < \omega} A_n \cup \bigcup_{n < \omega} \bar{a}_n$$

is a model.

Secondly we construct, by induction on $n < \omega$, tuples $\bar{b}_n$ such that

$$\text{Lstp}^w(\bar{b}_n\ldots\bar{b}_0/A) = \text{Lstp}^w(\bar{a}_n\ldots\bar{a}_0/A).$$

First let $\bar{b}_0 = \bar{b}$. Then 3.1 holds by assumption. Assume we have defined $\bar{b}_m$ for $m \leq n$. Let $\bar{b}_{n+1}$ be such that $\text{Lstp}(\bar{b}_{n+1}\bar{b}_n\ldots\bar{b}_0/A_{n+1}) = \text{Lstp}(\bar{a}_{n+1}\bar{a}_n\ldots\bar{a}_0/A_{n+1})$. We claim that 3.1 holds for $\bar{b}_{n+1}$. If not, there is finite $B \subset A$ such that

$$\text{Lstp}(\bar{b}_{n+1}\bar{b}_n\ldots\bar{b}_0/B) \neq \text{Lstp}(\bar{a}_{n+1}\bar{a}_n\ldots\bar{a}_0/B).$$

We may assume that $A_{n+1} \subset B$. By induction, $\text{Lstp}(\bar{a}_n\ldots\bar{a}_0/B) = \text{Lstp}(\bar{b}_n\ldots\bar{b}_0/B)$. Let $\bar{c}$ be such that

$$\text{Lstp}(\bar{c}\bar{a}_n\ldots\bar{a}_0/B) = \text{Lstp}(\bar{b}_{n+1}\bar{b}_n\ldots\bar{b}_0/B),$$
and hence also \( \text{Lstp}(\overline{c}a_n...a_0/A_{n+1}) = \text{Lstp}(\overline{b}_{n+1} \overline{b}_n...\overline{b}_0/A_{n+1}) = \text{Lstp}(\overline{a}_{n+1}a_n...a_0/A_{n+1}) \).

But now by isolation, also
\[
\text{Lstp}(\overline{c}a_n...a_0/B) = \text{Lstp}(\overline{a}_{n+1}a_n...a_0/B),
\]
and thus \( \text{Lstp}(\overline{b}_{n+1} \overline{b}_n...\overline{b}_0/B) = \text{Lstp}(\overline{a}_{n+1}a_n...a_0/B) \), a contradiction. Now we are done with the second construction.

For each finite \( A_n \subset A \), \( \text{Lstp}(\overline{b}_n...\overline{b}_0/A_n) = \text{Lstp}(\overline{a}_n...\overline{a}_0/A_n) \) implies that \( \text{tp}^g(\overline{b}_0...\overline{b}_n/A_n) = \text{tp}^g(\overline{a}_0...\overline{a}_n/A_n) \). There are automorphisms \( f_n \), for \( n < \omega \), witnessing this. By finite character of \( (K, \preceq_K) \), the mapping
\[
\bigcup_{n<\omega} f_n | (A_n \cup \overline{a}_0...\overline{a}_n) : \mathcal{B} \rightarrow \mathcal{M}
\]
extends to an automorphism \( f \in \text{Aut}(\mathcal{M}/A) \) such that \( f(\overline{a}) = f(\overline{a}_0) = \overline{b}_0 = \overline{b} \).

This proves the theorem. \( \square \)

We recall that \( (K, \preceq_K) \) is said to be \( \kappa \)-tame if for every model \( \mathcal{A} \) and tuples \( \overline{b} \) and \( \overline{a} \) such that
\[
\text{tp}^g(\overline{a}/\mathcal{A}) \neq \text{tp}^g(\overline{b}/\mathcal{A}),
\]
there is \( \mathcal{B} \preceq_K \mathcal{A} \) of size \( \leq \kappa \) such that
\[
\text{tp}^g(\overline{a}/\mathcal{B}) \neq \text{tp}^g(\overline{b}/\mathcal{B}).
\]

We say that \( (K, \preceq_K) \) is tame is it is \( \text{LS}(K) \)-tame. With tameness we can generalize the previous result to weak Lascar strong types over models of arbitrary size.

**Theorem 3.20.** Assume that \( (K, \preceq_K) \) is tame, simple, superstable, finitary AEC with the Tarski-Vaught -property. If \( \mathcal{A} \) is a model, then \( \text{Lstp}^w(\overline{a}/\mathcal{A}) = \text{Lstp}^w(\overline{b}/\mathcal{A}) \) implies that \( \text{tp}^g(\overline{a}/\mathcal{A}) = \text{tp}^g(\overline{b}/\mathcal{A}) \).

**Proof.** Assume that \( \mathcal{A} \) is a model and \( \text{Lstp}^w(\overline{a}/\mathcal{A}) = \text{Lstp}^w(\overline{b}/\mathcal{A}) \). Then also \( \text{Lstp}^w(\overline{a}/\mathcal{B}) = \text{Lstp}^w(\overline{b}/\mathcal{B}) \) for all countable \( \mathcal{B} \preceq_K \mathcal{A} \). Theorem 3.19 implies that \( \text{tp}^g(\overline{a}/\mathcal{B}) = \text{tp}^g(\overline{b}/\mathcal{B}) \) for all countable \( \mathcal{B} \preceq_K \mathcal{A} \). But now by tameness, \( \text{tp}^g(\overline{a}/\mathcal{A}) = \text{tp}^g(\overline{b}/\mathcal{A}) \). \( \square \)

**Theorem 3.21.** Assume that \( (K, \preceq_K) \) is tame, simple, superstable, finitary AEC with the Tarski-Vaught -property. If \( \mathcal{A} \) is an a-saturated model, then the following are equivalent:

1. \( \text{Lstp}^w(\overline{a}/\mathcal{A}) = \text{Lstp}^w(\overline{b}/\mathcal{A}) \)
2. \( \text{tp}^g(\overline{a}/\mathcal{A}) = \text{tp}^g(\overline{b}/\mathcal{A}) \)
3. \( \text{tp}^w(\overline{a}/\mathcal{A}) = \text{tp}^w(\overline{b}/\mathcal{A}) \).
Proof. By the previous theorem, (1) implies (2). Clearly (2) implies (3). It is enough to prove that equivalent weak types over $\mathcal{A}$ imply equivalent weak Lascar strong types over $\mathcal{A}$. Let $A \subset A$ be finite. We want to show that $Lstp(\bar{a}/A) = Lstp(\bar{b}/A)$. Since $A$ is a-saturated, there is $\bar{c} \in A$ realizing $Lstp(\bar{a}/A)$. Now since $tp^w(\bar{a}/\mathcal{A}) = tp^w(\bar{b}/\mathcal{A})$, there is $f \in Aut(M/A \cup \bar{c})$ such that $f(\bar{a}) = \bar{b}$. Then by invariance $Lstp(f(\bar{a})/A) = Lstp(f(\bar{c})/A)$, and thus $Lstp(\bar{b}/A) = Lstp(\bar{c}/A) = Lstp(\bar{a}/A)$. \[Q.E.D.\]

In section 3.3 we show that $\aleph_0$-stability implies superstability in simple finitary classes. Since $\aleph_0$-stability and finite character imply also the Tarski-Vaught property, we have that the above equivalence holds also in simple, tame $\aleph_0$-stable finitary classes. There we also have that all $\aleph_0$-saturated models are a-saturated, and hence have countable a-saturated models. The implication of Theorem 3.19 does not need tameness, and thus holds in $\aleph_0$-stable simple finitary classes.

By Corollary 3.4 and Theorem 3.21 we get the following.

**Theorem 3.22.** Assume that a simple, tame finitary $(K, \preceq_K)$ has the Tarski-Vaught property and is superstable. Then it is Galois-stable in each cardinal $\mu \geq L(K)$, where $L(K)$ is the least upper bound for the number of Lascar strong types over a finite set.

**3.3. Characterization of superstability.** In this section we study how the concepts of superstability, weak superstability and $\aleph_0$-stability are related. We show that a nice behaviour weak Lascar strong types implies our notions of weak superstability and superstability being equivalent (Corollary 3.27) and that simple $\aleph_0$-stable finitary classes are superstable (Corollary 3.28). We also characterize superstability with some equivalent conditions in Theorem 3.29.

We define a temporary notion called dominating weak Lascar strong types. We say that the class $(K, \preceq_K)$ has $\lambda$-dominating weak Lascar strong types, if for every model $\mathcal{A}$ of size $\leq \lambda$ and tuples $\bar{a}$ and $\bar{b}$, whenever $Lstp^w(\bar{a}/\mathcal{A}) = Lstp^w(\bar{b}/\mathcal{A})$, then $tp^\mathcal{A}(\bar{a}/\mathcal{A}) = tp^\mathcal{A}(\bar{b}/\mathcal{A})$. We say that the class has dominating weak Lascar strong types if it has $\lambda$-dominating weak Lascar strong types for all $\lambda$. Tameness, Tarski-Vaught property and superstability imply dominating weak Lascar strong types in a simple, finitary $(K, \preceq_K)$ by Theorem 3.20. We will show that also weak superstability and dominating weak Lascar strong types imply superstability.

**Lemma 3.23.** Assume that a simple finitary $(K, \preceq_K)$ has $\lambda$-dominating weak Lascar strong types. Assume that $\bar{a} \downarrow_A B \cup A$, where $B$ is a model, $|B| \leq \lambda$ and $A$ is a finite set, not necessarily a subset of $B$. Assume also that $B \subseteq C$, where $C$ is a set. Then there is $g \in Aut(M/A \cup B)$ such that $g(\bar{a}) \downarrow_A C$.

**Proof.** By Corollary 2.12 there is $\bar{b}$ realizing $Lstp(\bar{a}/A)$ such that $\bar{b} \downarrow_A A \cup C$. We write the finite set $A$ as a sequence $a'$. Now by stationarity (Proposition...
2.30, Lstp\(^w\)(\(\bar{a}/\mathcal{B} \cup A\)) = Lstp\(^w\)(\(\bar{b}/\mathcal{B} \cup A\)), and furthermore Lstp\(^w\)(\(\bar{a}^w/\mathcal{B}\)) = Lstp\(^w\)(\(\bar{b}^w/\mathcal{B}\)). Then by \(\lambda\)-dominating weak Lascar strong types, there is \(g \in \text{Aut}(\mathcal{M}/\mathcal{B})\) such that \(g(\bar{a}^w) = \bar{b}^w\). Hence \(g(\bar{a}) \downarrow_A A \cup C\) and \(g \in \text{Aut}(\mathcal{A}/\mathcal{B} \cup A)\).

\[\Box\]

**Lemma 3.24.** Assume that a simple, finitary, weakly stable \((\mathbb{K}, \preceq_\mathbb{K})\) has \(\lambda\)-dominating weak Lascar strong types. Let \(A \subset \mathcal{B}\), where \(A\) is finite, \(\mathcal{B}\) is a model of size \(\leq \lambda\) and let \(\alpha\) be an ordinal. If \(\bar{a} \downarrow_A \mathcal{B}\), then there is a strongly \(\mathcal{B}\)-indiscernible sequence \((\bar{a}_i)_{i<\alpha}\) such that \(\bar{a}_0 = \bar{a}\) and

\[\bar{a}_i \downarrow_A \mathcal{B} \cup \bigcup_{j<i} \bar{a}_j\] for each \(i < \alpha\).

**Proof.** If \(\text{tp}^w(\bar{a}/A)\) is bounded, we can take the trivial sequence. Thus we may assume that \(\text{tp}^w(\bar{a}/A)\) is unbounded. Using Proposition 2.12 we define \((\bar{b}_i)_{i<\text{H}(|\mathcal{B}|)}\) such that each \(\bar{b}_i\) realizes \(\text{Lstp}(\bar{a}/A)\) and

\[\bar{b}_i \downarrow_A \mathcal{B} \cup \bigcup_{j<i} \bar{b}_j\] for each \(i < \text{H}(|\mathcal{B}|)\).

By stationarity of weak Lascar strong types, each \(\bar{b}_i\) realizes \(\text{Lstp}^w(\bar{a}/\mathcal{B})\) and by \(\lambda\)-dominating weak Lascar strong types, also \(\text{tp}^g(\bar{a}/\mathcal{B})\).

For each \(i < \text{H}(|\mathcal{B}|)\), the type \(\text{tp}^w(\bar{b}_i/A) = \text{tp}^w(\bar{a}/A)\) is unbounded, and thus by Lemma 2.9(2) these \(\bar{b}_i\) are distinct. We can use Lemma 2.4 to find strongly \(\mathcal{B}\)-indiscernible sequence \((\bar{a}_i)_{i<\alpha}\) such that for each \(n < \omega\) and \(j_0 < ... < j_n < \alpha\) there are \(i_0 < ... < i_n < \text{H}(|\mathcal{B}|)\) such that

\[\text{tp}^g(\bar{a}_{j_0}, ..., \bar{a}_{j_n}/\mathcal{B}) = \text{tp}^g(\bar{b}_{i_0}, ..., \bar{b}_{i_n}/\mathcal{B}).\]

Hence by finite character, \(\bar{a}_i \downarrow_A \mathcal{B} \cup \bigcup_{j<i} \bar{a}_j\) for each \(n < \omega\).

Now \(\bar{a}_0\) realizes \(\text{tp}^g(\bar{a}/\mathcal{B})\) and thus there is an automorphism \(f \in \text{Aut}(\mathcal{M}/\mathcal{B})\) mapping \(\bar{a}_0\) to \(\bar{a}\). We may assume that \(\bar{a}_0 = \bar{a}\).

In the following lemma we use again the finite character of \((\mathbb{K}, \preceq_\mathbb{K})\).

**Lemma 3.25.** Assume that a simple finitary weakly stable \((\mathbb{K}, \preceq_\mathbb{K})\) has \(\lambda\)-dominating weak Lascar strong types. Let \((A_k)_{k<\omega}\) be an increasing sequence of finite sets such that \(\bigcup_{k<\omega} A_k\) is a model, \(A_2 \cup \bar{a} \downarrow_{A_1} \mathcal{C}\) and \(A_{k+1} \downarrow_{A_k \cup \mathcal{C}} \mathcal{C}\) for \(k \geq 2\). Assume also that \(\mathcal{C}\) is a model of size \(\leq \lambda\) and \(\mathcal{C} \subset D\). Then there is an increasing sequence of finite sets \((B_k)_{k<\omega}\) and finite \(\bar{b}\) such that

\[
\begin{align*}
(1) & \quad \text{tp}^g(B_k \cup \bar{b}/A_1 \cup \mathcal{C}) = \text{tp}^g(A_k \cup \bar{a}/A_1 \cup \mathcal{C}), \\
(2) & \quad B_2 \cup \bar{b} \downarrow_{A_1} D, \\
(3) & \quad B_{k+1} \downarrow_{B_k \cup \mathcal{C}} D \text{ for } k \geq 2
\end{align*}
\]
(4) $\bigcup_{k<\omega} B_k$ is a model.

Proof. Since $A_2 \cup \bar{a} \downarrow A_1 \mathcal{C}$, by Lemma 3.23 there is $g_2 \in \text{Aut}(\mathcal{M}/A_1 \cup \mathcal{C})$ such that $g_2(A_2 \cup \bar{a}) \downarrow A_1 D$. We let $\bar{b} = g_2(\bar{a})$ and $B_2 = g_2(A_2)$. Since $A_3 \downarrow A_2 \cup \bar{a} \mathcal{C}$, by invariance also $g_2(A_3) \downarrow B_2 \cup \bar{b} \mathcal{C}$ and again by Lemma 3.23, there is $g \in \text{Aut}(\mathcal{M}/B_2 \cup \bar{b} \cup \mathcal{C})$ such that $g(g_2(A_3)) \downarrow B_2 \cup \bar{b} D$. We let $g_3 = g \circ g_2$ and $B_3 = g_3(A_3)$.

Let $k \geq 3$ and assume we have defined $g_k \in \text{Aut}(\mathcal{M}/g_{k-1}(A_{k-1} \cup \bar{a}))$ such that $g_{k-1}(\bar{a}) = \bar{b}$, $g_k \uparrow (A_1 \cup \mathcal{C}) = \text{Id}(A_1 \cup \mathcal{C})$ and $g_k(A_k) \downarrow g_{k-1}(A_{k-1} \cup \bar{a}) D$. We denote $B_k = g_k(A_k)$.

Since $A_{k+1} \downarrow A_k \cup \bar{a} \mathcal{C}$, also $g_k(A_{k+1}) \downarrow g_k(A_k \cup \bar{a}) \cup \mathcal{C}$ and thus by Lemma 3.23, there is $g \in \text{Aut}(\mathcal{M}/g_k(A_k \cup \bar{a}) \cup \mathcal{C})$ such that $g(g_k(A_{k+1})) \downarrow g_k(A_k \cup \bar{a}) D$. We let $g_{k+1} = g \circ g_k$. Now, when $B_{k+1} = g_{k+1}(A_{k+1})$, we have that $\text{tp}^g(B_k \cup \bar{b} \cup A_1 \cup \mathcal{C}) = \text{tp}^g(A_k \cup \bar{a} \cup A_1 \cup \mathcal{C})$ and $B_{k+1} \downarrow B_k \cup \bar{b} D$ when $k \geq 2$.

Finally the mapping $\bigcup_{k<\omega} g_k \downarrow A_k : \bigcup_{k<\omega} A_k \rightarrow \bigcup_{k<\omega} B_k$ preserves Galois types of finite tuples, and thus by finite character of $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a $\mathbb{K}$-embedding. We get that $\bigcup_{k<\omega} B_k$ is a model. \qed

Proposition 3.26. Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple, finitary AEC. Assume also that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is stable in $\lambda$ and has $\lambda$-dominating weak Lascar strong types for some $\lambda$ such that $\lambda^{\mathbb{K}_0} > \lambda$. Let $(A_k)_{k<\omega}$ be an increasing sequence of finite sets such that $\bigcup_{k<\omega} A_k$ is a model and let $\bar{a} \uparrow$ be a tuple. Then there is $k < \omega$ such that $\bar{a} \downarrow A_{k+1}$.

Proof. Assume the contrary. Let $(A_k)_{k<\omega}$ be an increasing sequence of finite sets such that $\bigcup_{k<\omega} A_k$ is a model and $\bar{a} \notin A_{k+1}$ for all $k < \omega$.

For each mapping $\xi : \omega \rightarrow \lambda$ and $k, n < \omega$, we define finite $A^{k}_{\xi|n}$, $\bar{a}_{\xi|n}$ and a set $\mathcal{A}_{n}$, such that

1. $\mathcal{A}_{n} \subset \mathcal{A}_{n+1}$ and $|\mathcal{A}_{n}| \leq \lambda$ for each $n < \omega$,
2. $\mathcal{A}_{0} = A_{0}$ is finite but $\mathcal{A}_{n}$ is a model for each $0 < n < \omega$,
3. $\bigcup_{n\rightarrow\lambda} A^{n}_{\xi|n} \subset \mathcal{A}_{n}$,
4. When $\xi \uparrow \xi' \uparrow n$ and $\xi'(n) < \xi(n)$, $\text{tp}^g(\bar{a}_{\xi|n+1}/A^{n+1}_{\xi'|n+1}) \neq \text{tp}^g(\bar{a}_{\xi'|n+1}/A^{n+1}_{\xi'|n+1})$
   and
5. for $m < n < \omega$, $\text{tp}^g(\bar{a}_{\xi|m}/\mathcal{A}_{m}) = \text{tp}^g(\bar{a}_{\xi|m}/\mathcal{A}_{m})$.

Then by Lemma 3.6, for each $\xi : \omega \rightarrow \lambda$, we will gain $\bar{a}_{\xi}$ satisfying $\text{tp}^g(\bar{a}_{\xi|n}/\mathcal{A}_{n})$ for each $n < \omega$. By (4), these $\bar{a}_{\xi}$ will contradict $\lambda$-stability. We do the construction maintaining the following three conditions.

(i) We have $A^{k}_{\xi|n} \subset A^{k+1}_{\xi|n}$ for each $k < \omega$ and $\bigcup_{k<\omega} A^{k}_{\xi|n}$ is a model.
(ii) For each $\xi : \omega \rightarrow \lambda$ and $n < \omega$
   (a) $\xi' \uparrow n = \xi \uparrow n$ and $\xi'(n) < \xi(n)$ imply $A^{n+1}_{\xi|n+1} \downarrow A^{n+1}_{\xi'|n+1}$.
(b) \((A^{n+1}_{\xi|n+1})_{\xi(n)<\lambda}\) is a strongly \(\mathcal{A}_n\)-indiscernible sequence with \(A^{n+1}_{\xi|n+1} = A^{n+1}_{\xi|n}\) for \(\xi(n) = 0\).

(c) For each \(n < \omega\), \(j < \lambda\) there is \(F^{n+1}_j \in \text{Aut}(\mathcal{M}/\mathcal{A}_n)\) such that \(F^{n+1}_j(A^{n+1}_{\xi|n}) = A^{n+1}_{\xi|n+1}\) for \(\xi\) such that \(\xi(n) = j\).

(d) The model \(\mathcal{A}_{n+1}\), is closed under the functions \(F^{n+1}_j\) and their inverses for \(j < \lambda\).

(iii) When \(k \geq n\), \(A^{k+1}_{\xi|n} \downarrow A^{k}_{\xi|n} \cup \bar{a}_{\xi|n} \dashv A^{k+1}_{\xi|n}\) and \(A^{n+1}_{\xi|n} \cup \bar{a}_{\xi|n} \downarrow A^{n}_{\xi|n} \mathcal{A}_n\).

First let \(A^{k}_{\xi|0} = A_k\) for each \(k < \omega\), \(\bar{a}_{\xi|0} = \bar{a}\) and \(\mathcal{A}_0 = A_0 = A^{0}_{\xi|0}\). Then (ii) holds trivially, and (i) and (iii) hold by simplicity, monotonicity and the assumption. Also (1)-(5) hold trivially.

Assume we have defined everything for \(m \leq n\). Since \(A^{n+1}_{\xi|n} \downarrow A^{n}_{\xi|n} \mathcal{A}_n\) by (iii), we can use (Proposition 2.31 or) Lemma 3.24 to find a strongly \(\mathcal{A}_n\)-indiscernible \((A^{n+1}_{\xi|n+1})_{\xi(n)<\lambda}\) such that \(A^{n+1}_{\xi|n+1} = A^{n+1}_{\xi|n}\) for \(\xi(n) = 0\) and (ii)(a) holds. Then define \(F^{n+1}_j\) and \(\mathcal{A}_{n+1}\) as wanted, \(F^{n+1}_0\) being the identity. We require that \(\mathcal{A}_n \cup \bigcup_{\xi(n+1)-\lambda} A^{n+1}_{\xi|n+1} \subset \mathcal{A}_{n+1}\), \(|\mathcal{A}_{n+1}| \leq \lambda\) and \(\mathcal{A}_{n+1}\) is closed under each \(F^{n+1}_j\) and \((F^{n+1}_j)^{-1}\).

By (iii), we have \(A^{n+2}_{\xi|n} \downarrow A^{n+1}_{\xi|n} \cup \bar{a}_{\xi|n} \downarrow A^{n+1}_{\xi|n} \mathcal{A}_n\) and \(A^{n+1}_{\xi|n} \cup \bar{a}_{\xi|n} \downarrow A^{n}_{\xi|n} \mathcal{A}_n\). We gain by Pairs lemma and left transitivity that

\[A^{n+2}_{\xi|n} \cup \bar{a}_{\xi|n} \downarrow A^{n+1}_{\xi|n} \mathcal{A}_n.\]

On the other hand, we have \(A^{k+1}_{\xi|n} \downarrow A^{k}_{\xi|n} \cup \bar{a}_{\xi|n} \dashv A^{k+1}_{\xi|n}\) and \(\bar{a}_{\xi|n} \dashv A^{k+1}_{\xi|n}\) for each \(k \geq n+1\). By Lemma 3.25, there are \(\bar{b}\) and finite \(B_k\) for \(n + 1 \leq k < \omega\), such that \(B_{n+1} = A^{n+1}_{\xi|n}\) and for each \(k \geq n + 1\),

(a) \(B_k \cup \bar{b}\) realizes \(tp^\mathcal{A}(A^{k}_{\xi|n} \cup \bar{a}_{\xi|n}/A^{n+1}_{\xi|n} \cup \mathcal{A}_n)\) and thus \(\bar{b} \not\equiv_{B_k} B_{k+1}\).

(b) \(B_{n+2} \cup \bar{b} \downarrow A^{n+1}_{\xi|n} \mathcal{A}_{n+1}\),

(c) \(B_{k+1} \not\equiv_{B_k \cup \bar{b}} \mathcal{A}_{n+1}\) when \(k \geq n + 2\) and

(d) \(B_k \subset B_{k+1}\) and \(\bigcup_{k<\omega} B_k\) is a model.

Since \(B_{n+1} = A^{n+1}_{\xi|n}\), we have that

\[F^{n+1}_{\xi(n)}(B_{n+1}) = A^{n+1}_{\xi|n+1}.\]

Define for each \(\xi(n) < \lambda\) and \(n + 1 < k < \omega\),

\[\bar{a}_{\xi|n+1} = F^{n+1}_{\xi(n)}(\bar{b})\text{ and} A^{k}_{\xi|n+1} = F^{n+1}_{\xi(n)}(B_k).\]
Since each $F_{j+1}^{n+1}$ maps $\mathcal{A}_{n+1}$ to itself, we get from (b1)-(b3) that (iii) holds for $n+1$. Also by (b4), (i) holds. We check that a (1)-(5) hold. Items (1)-(3) hold by the definition of $\mathcal{A}_{n+1}$. Also (5) holds, since $\bar{b}$ realizes $tp^g(\bar{a}_\xi|_n/\mathcal{A}_n)$ and $F_{j+1}^{n+1} \in \text{Aut}(\mathcal{M}/\mathcal{A}_n)$ for each $j < \lambda$. We claim that (4) holds.

Let $\xi' \upharpoonright n = \xi \upharpoonright n$ and $\xi'(n) < \xi(n)$. Since $\bar{a}_{\xi|_{n+1}} \downarrow_{A_{\xi|_{n+1}}}^n \mathcal{A}_{n+1}$ by (iii) and $A_{\xi'|_{n+1}}^{n+1} \subset \mathcal{A}_{n+1}$, we have that $\bar{a}_{\xi'|_{n+1}} \downarrow_{A_{\xi'|_{n+1}}}^n \mathcal{A}_{n+1}$. Also $A_{\xi|_{n+1}}^n \downarrow_{A_{\xi|_{n}}}^n \mathcal{A}_{\xi'|_{n+1}}^{n+1}$ by (ii). We get from Pairs lemma and monotonicity that

\[
\bar{a}_{\xi|_{n+1}} \downarrow_{A_{\xi|_{n}}}^n \mathcal{A}_{\xi'|_{n+1}}^{n+1}.
\]

On the other hand, $\bar{a}_{\xi|_{n}} \not\downarrow_{A_{\xi|_{n}}}^n \mathcal{A}_{\xi|_{n}}^{n+1}$ by (iii), and thus $\bar{b} \upharpoonright A_{\xi|_{n}}^n \mathcal{A}_{\xi|_{n}}^{n+1}$ by (b1). The automorphism $F_{\xi'(n)}^{n+1}$ gives that

\[
\bar{a}_{\xi'|_{n+1}} \upharpoonright A_{\xi|_{n}}^n \mathcal{A}_{\xi'|_{n+1}}^{n+1}.
\]

We get that $tp^g(\bar{a}_{\xi|_{n+1}}/\mathcal{A}_{\xi'|_{n+1}}^{n+1}) \neq tp^g(\bar{a}_{\xi'|_{n+1}}/\mathcal{A}_{\xi'|_{n+1}}^{n+1})$. \hfill \Box

From the previous proposition we get two important corollaries.

**Corollary 3.27.** Assume that $(K, \preceq_K)$ is a simple finitary AEC with dominating weak Lascar strong types. Then $(K, \preceq_K)$ is superstable if and only if it is weakly superstable.

*Proof.* The other direction is Corollary 3.4. If $(K, \preceq_K)$ is weakly superstable and has dominating weak Lascar strong types, we can clearly find the required $\lambda$ for Proposition 3.26. \hfill \Box

**Corollary 3.28.** Assume that $(K, \preceq_K)$ is an $\aleph_0$-stable simple finitary AEC. Then it is superstable.

*Proof.* By Theorem 3.12 of [8], in $\aleph_0$-stable finitary AEC, equivalence of weak types implies equivalence of Galois types over countable models. Thus any $\aleph_0$-stable finitary AEC has $\aleph_0$-dominating weak Lascar strong types. Superstability follows by Proposition 3.26. \hfill \Box

Finally we give a list of properties equivalent to superstability.

**Theorem 3.29 (Characterization of superstability).** Let $(K, \preceq_K)$ be a simple finitary AEC with the Tarski-Vaught-property. The following are equivalent.

1. The class $(K, \preceq_K)$ is weakly stable and for all finite and increasing $A_n$, $n < \omega$ and $\bar{a}$ there is $n < \omega$ such that $\bar{a} \downarrow_{A_n} A_{n+1}$.
(2) **Superstability:** The class \((K, \preceq_K)\) is weakly stable and for all finite and increasing \(A_n, n < \omega\) such that \(\bigcup_{n<\omega} A_n\) is a model and \(\bar{a}\) there is \(n < \omega\) such that \(\bar{a} \downarrow_{A_n} A_{n+1}\).

(3) The class \((K, \preceq_K)\) is weakly superstable and there is an infinite cardinal \(\kappa\) such that for any \(\bar{a}\) and finite \(A\), there is a strongly \(A\)-indiscernible \((\bar{a}_i)_{i<\kappa}\) such that for any \(\bar{b}\),

\[|\{i < \kappa : \bar{b} \not\models A \bar{a}_i\}| < \kappa.\]

(4) There are infinite cardinals \(\lambda\) and \(\kappa\) such that \(\kappa^{\aleph_0} \leq \lambda\), \(\lambda^{\aleph_0} > \lambda\), \((K, \preceq_K)\) is weakly stable in \(\lambda\) and for any \(\bar{a}\) and finite \(A\), there is a strongly \(A\)-indiscernible \((\bar{a}_i)_{i<\kappa}\) such that for any \(\bar{b}\),

\[|\{i < \kappa : \bar{b} \not\models A \bar{a}_i\}| < \kappa.\]

If the class is also tame, (1)-(4) are equivalent to

(5) The class \((K, \preceq_K)\) is weakly superstable and whenever a finite tuple \(\bar{b}\) realizes \(\text{Lstp}^w(\bar{a}/A)\), where \(A\) is a model, there is \(f \in \text{Aut}(M/A)\) such that \(f(\bar{b}) = \bar{a}\).

**Proof.** Items (1) and (2) are equivalent by Proposition 3.11. By Lemma 3.7, (4) implies (2). Clearly also (3) implies (4). We show that (1) implies (3), and then we are done with the first part of the theorem. By Corollary 3.4, (1) implies weak superstability. To prove (3), let \(\bar{a}\) and \(A\) be finite. We prove (3) for the infinite cardinal \(\kappa = \aleph_0\). By Lemma 2.31, there is a strongly \(A\)-indiscernible sequence \((\bar{a}_n)_{n<\omega}\) such that \(\bar{a}_0 = \bar{a}\) and

\[\bar{a}_n \downarrow_A \bigcup_{m<n} \bar{a}_m\text{ for each }n < \omega.\]

We claim that this is the required sequence. We assume the contrary, that there would be some \(\bar{b}\) such that \(\bar{b} \not\models A \bar{a}_n\) for infinitely many \(n < \omega\). Let \((\bar{b}_n)_{n<\omega}\) be this infinite subsequence. Then we claim that

\[\bar{b} \not\models A \cup \bigcup_{m<n} \bar{b}_m \text{ for each }n < \omega.\]

To prove this second claim, again assume the contrary that \(\bar{b} \downarrow_{A \cup \bigcup_{m<n} \bar{b}_m} \bar{b}_n\) for some \(n\). But now \(\bar{b}_n \downarrow_A \bigcup_{m<n} \bar{b}_m\) by the definition of the sequence, and we get by symmetry and transitivity that \(\bar{b}_n \downarrow_A \bar{b} \cup \bigcup_{m<n} \bar{b}_m\). Then \(\bar{b} \not\models A \bar{b}_n\) by monotonicity and symmetry, a contradiction. This proves the second claim. To prove the first claim we define increasing and finite sets \(A_n := A \cup \bigcup_{m<n} \bar{b}_m\). Now \(\bar{b} \not\models A_{n+1}\) for each \(n < \omega\), a contradiction with (1).
Item (5) follows from (2) by Corollary 3.4 and Theorem 3.20, where we need tameness. Item (2) follows from (5) by Proposition 3.26, since weak superstability clearly implies weak stability.

We note that in the previous theorem implication from (4) to (2) holds also without the Tarski-Vaught-property.

3.4. a-categoricity. One of the basic results for abstract elementary classes with amalgamation, joint embedding and arbitrarily large models, shown by Shelah, is that categoricity in any uncountable cardinal implies stability in $\text{LS}(\mathbb{K})$. We also proved in [8] that in our case stability in $\text{LS}(\mathbb{K}) = \aleph_0$ implies weak stability in each infinite cardinal. Since we now want to study the case without $\aleph_0$-stability, we will consider a weakening of categoricity called a-categoricity, and study when a-categoricity implies superstability. We recall that a model is said to be a-saturated, if every Lascar strong type over finite subset is realized in the model.

**Definition 3.30 (a-categoricity).** We say that the class $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a-categorical in $\kappa$ if there is exactly one a-saturated model of size $\kappa$, up to isomorphism.

Let us denote by $L(\mathbb{K})$ the supremum of the number of Lascar strong types over any finite set. For any single finite set $E$, we know by Lemma 2.4, that the number of Lascar strong types over this set is strictly less than the number $H = \beth_{(2^{\aleph_0})^+}$. To count the value of $L(\mathbb{K})$, we should take the supremum over each finite subset in the monster model $\mathcal{M}$. On the other hand, if there is an automorphism $f \in \text{Aut}(\mathcal{M})$ mapping a finite set $E_1$ to another finite set $E_2$, there is exactly the same number of Lascar strong types over $E_1$ and $E_2$. Since there are at most $2^{\aleph_0}$ different isomorphism types of countable structures in $\mathbb{K}$, there are at most $\aleph_0$ times $2^{\aleph_0}$ many finite sets in the monster model, up to automorphism. Now since $cf(H) > 2^{\aleph_0}$, we get that

$$L(\mathbb{K}) < H.$$

There exists an a-saturated model in every cardinal $\kappa \geq L(\mathbb{K})$.

When $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a finitary abstract elementary class, we can study the class $(\mathbb{K}_a, \preceq_{\mathbb{K}})$, where $\mathbb{K}_a$ is the class of a-saturated models of $\mathbb{K}$. The class $(\mathbb{K}_a, \preceq_{\mathbb{K}})$ is an abstract elementary class with amalgamation, joint embedding, arbitrarily large models and $\text{LS}(\mathbb{K}_a) = L(\mathbb{K})$. Also $\text{LS}(\mathbb{K})$-tameness of $(\mathbb{K}, \preceq_{\mathbb{K}})$ implies $\text{LS}(\mathbb{K}_a)$-tameness for $(\mathbb{K}_a, \preceq_{\mathbb{K}})$. Many results from more general theory of abstract elementary classes can be adapted for $(\mathbb{K}_a, \preceq_{\mathbb{K}})$. Also a-categoricity transfer follows for certain cardinals, see Theorem 4.12.

We state the following results which adapt the presentation theorem and the construction of Ehrenfeucht-Mostowski models for AEC by Shelah.
Proposition 3.31. There is a class \( \mathcal{K}^* \) of \( \tau^* \)-structures with \( \tau^* = \tau \cup \{ F^k_i : k < \omega, i < L(\mathcal{K}) \} \), where each \( F^k_i \) is a \( k \)-ary function symbol and the following holds

1. If \( \mathcal{A}^* \in \mathcal{K}^* \) and \( B \subset \mathcal{A}^* \) a subset such that \( B \) is closed under functions \( F^k_i \), then \( B \) is an \( \mathcal{A}^* \)-saturated \( \mathcal{K}^* \)-elementary substructure of \( \mathcal{A}^* \mid \tau \).
2. For every \( \mathcal{A}^* \subset \mathcal{K}^* \) there is \( \mathcal{A}^* \in \mathcal{K}^* \) such that \( \mathcal{A}^* \mid \tau = \mathcal{A} \).

When \( \mathcal{A}^* \in \mathcal{K}^* \) and \( A \subset \mathcal{A}^* \), we denote as \( SH(A) \) the closure of \( A \) under the functions \( F^k_i \), \( k < \omega, i < L(\mathcal{K}) \). By the previous theorem, \( SH(A) \) is an \( \mathcal{A}^* \)-saturated \( \mathcal{K}^* \)-elementary substructure of \( \mathcal{A} \mid \tau \). The following formulation of the Ehrenfeucht-Mostowski model construction is tailored for the purposes of this paper. The proof of this theorem is similar than the proof of Proposition 2.13 in [7]. First we recall the concept of a tidy sequence from [7].

Definition 3.32 (Tidy sequence). Let \( i_0^\alpha < \ldots < i_n^\alpha \in I \) for each \( \alpha < \lambda \), where \( I \) is a linear order. We say that the sequence \( (i_0^\alpha, \ldots, i_n^\alpha)_{\alpha < \lambda} \) is tidy, if for each \( 0 \leq k \leq n \) one of the following holds.

1. The index at \( k \) is constant, that is, \( i_k^\alpha = \beta \in I \) is fixed for each \( \alpha < \lambda \).
2. The index at \( k \) is included in some \((m+1)\)-block, that is, \( k \in \{ p, p+1, \ldots, p+m \} \) such that
   
   a. \( p+m+1 > n \) or for each \( \beta < \lambda \), we have \( i_{p+m+1}^\beta \geq \sup \{ i_{p+m}^\alpha : \alpha < \lambda \} \),
   b. \( p-1 < 0 \) or for each \( \beta < \lambda \), we have \( i_{p-1}^\beta \leq \min \{ i_p^\alpha : \alpha < \lambda \} \) and
   c. for each \( \alpha < \beta < \lambda \), we have \( i_p^\alpha < \ldots < i_{p+m}^\alpha \).

Proposition 3.33. For any linear order \( I \) and set \( A \) there is a sequence \( (a_i)_{i \in I} \) and a model \( EM(I, A) \in \mathcal{K}^* \) with \( A \cup (a_i)_{i \in I} \subset EM(I, A) \) such that

1. \( |EM(I, A)| = |I| + |A| + L(\mathcal{K}) \),
2. Each element in \( EM(I, A) \) is a \( \tau^* \)-term from some \( a_{i_0}, \ldots, a_{i_n} \) and \( \bar{a} \) with \( n < \omega \), \( i_0 < \ldots < i_n \in I \) and \( \bar{a} \in A \).
3. Each partial order-preserving \( f : I \rightarrow I \) extends to an \( \tau^* \)-isomorphism \( F : SH(\{ a_i : i \in \text{dom}(f) \} \cup A) \rightarrow SH(\{ a_i : i \in \text{rng}(f) \} \cup A) \) mapping \( a_i \) to \( a_{f(i)} \) for each \( i \in \text{dom}(f) \) and fixing \( A \) pointwise.
4. Let \( (i_0^\alpha, \ldots, i_n^\alpha)_{\alpha < \lambda} \), \( i_0^\alpha < \ldots < i_n^\alpha \in I \), be a tidy sequence and let
   \( \bar{b}_i = \bar{f}(a_{i_0^\alpha}, \ldots, a_{i_n^\alpha}) \)
   for a fixed sequence \( \bar{f} \) of terms of \( \tau^* \). Then \( (\bar{b}_i)_{i < \alpha} \) is a strongly \( A \)-indiscernible sequence.

We identify \( (a_i)_{i \in I} \) with \( I \). The proof of the following Theorem is standard, using Ehrenfeucht-Mostowski models.
Theorem 3.34. (Shelah) Let \((\mathbb{K} \preceq \mathbb{K})\) be an AEC with amalgamation, joint embedding, arbitrarily large models and \(LS(\mathbb{K}) \leq L(\mathbb{K})\). Assume that \((\mathbb{K}, \preceq)\) is \(a\)-categorical in some \(\kappa > L(\mathbb{K})\) and let \(\kappa > \mu \geq L(\mathbb{K})\). Then \((\mathbb{K}, \preceq)\) is Galois-stable in \(\mu\).

From the previous theorem it follows that when a simple finitary AEC \((\mathbb{K}, \preceq)\) is \(a\)-categorical for some \(\kappa > L(\mathbb{K})\), it is weakly stable, and we can use the restricted properties of \(\downarrow\) studied in section 2.2. We also get the following corollary as usual.

Corollary 3.35. Let \((\mathbb{K} \preceq \mathbb{K})\) be an AEC with amalgamation, joint embedding, arbitrarily large models and \(LS(\mathbb{K}) \leq L(\mathbb{K})\). Assume that \((\mathbb{K}, \preceq)\) is \(a\)-categorical in some \(\kappa > L(\mathbb{K})\) and let \(\mu\) be such that \(cf(\kappa) \geq \mu\). Then the categorical \(a\)-saturated model of size \(\kappa\) is \(\mu\)-saturated respect to Galois types.

The next lemma gives another property that we want the \(a\)-categorical model to have.

Lemma 3.36. Assume that \((\mathbb{K}, \preceq)\) is a simple finitary AEC. Let \(\lambda > L(\mathbb{K})\). There is an \(a\)-saturated model \(A\) of size \(\lambda\) with the following property:

For any \(A \subset \mathcal{A}\) such that \(|A| \leq L(\mathbb{K})\), finite \(B \subset A\) and tuple \(\bar{a}\) there is \(\bar{b} \in \mathcal{A}\) realizing \(Lstp(\bar{a}/B)\) such that \(\bar{b} \downarrow_B A\).

Proof. We construct the model \(\mathcal{A}\) as an increasing and continuous union of \(a\)-saturated models \(\mathcal{A}_i\) of size \(\lambda\), for \(i < L(\mathbb{K})^+\), such that the following holds: for any finite \(B \subset \mathcal{A}_i\) and tuple \(\bar{a}\) there is \(\bar{b} \in \mathcal{A}_{i+1}\) realizing \(Lstp(\bar{a}/B)\) such that \(\bar{b} \downarrow_B \mathcal{A}_i\). Then since \(L(\mathbb{K})^+\) is regular, for any \(A \subset \mathcal{A}\) such that \(|A| \leq L(\mathbb{K})\) we can find \(i < L(\mathbb{K})^+\) such that \(A \subset \mathcal{A}_i\). We see that the model \(\mathcal{A}\) is as wanted by monotonicity.

We do the construction as follows. First let \(\mathcal{A}_0\) be any \(a\)-saturated model of size \(\lambda\). At the limit step we take union, so it is enough to construct the model in each successor step. Assume we have defined \(\mathcal{A}_i\). Let \((B_j)_{j < \lambda}\) enumerate all finite subsets of \(\mathcal{A}_i\). Then let \((\bar{a}_j^k)_{k < L(\mathbb{K})}\) enumerate representatives for each Lascar strong type over a set \(B_j\). For any finite \(B_i \subset \mathcal{A}_i\) and a tuple \(\bar{a}_i^k\), there is some \(\bar{b}_k \in \mathcal{A}_{i+1}\) realizing \(Lstp(\bar{a}_i^k/B_j)\) such that \(\bar{b}_k \downarrow_B \mathcal{A}_i\). This follows from simplicity and Corollary 2.12. Now let \(\mathcal{A}_{i+1}\) be an \(a\)-saturated model of size \(\lambda\) containing

\[
\mathcal{A}_i \cup \bigcup_{j < \lambda, k < L(\mathbb{K})} \bar{b}_k^j.
\]

We are done with the construction. \(\square\)

In the following proposition we assume that the \(a\)-categoricity cardinal has uncountable cofinality. This is needed to ensure that the categorical \(a\)-saturated model
satisfies all weak types over countable subsets. Hence we are only able to prove superstability from a-categoricity in a cardinal with uncountable cofinality. This is a flaw also in our a-categoricity transfer theorem, and we would like to know whether it is possible to drop this assumption.

**Proposition 3.37.** Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be a simple finitary AEC. Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a-categorical in \(\kappa > L(\mathbb{K})\) with uncountable cofinality. For each \(\bar{a}\) and finite \(A\) there is a strongly \(\tau\)-indiscernible sequence \((\bar{a}_i)_{i<\omega}\) such that \(\bar{a}_0 = \bar{a}\) and for any \(\bar{b}\), the set \(\{i < \omega : \bar{b} \not\equiv_{\bar{A}} \bar{a}_i\}\) is finite.

**Proof.** Let \(I = \mathbb{Q} + \kappa + \omega\). The model \(EM(I, A)\) has size \(\kappa\) and thus is the one \(a\)-saturated model of size \(\kappa\). This model has the property of Lemma 3.36 and is \(\aleph_1\)-saturated by Corollary 3.35. It is enough to study any \(\bar{a}'\) realizing \(tp^w(\bar{a}/A)\), and hence, by Lemma 3.36, we may assume that \(\bar{a} \subseteq EM(I, A)\) and \(\bar{a} \downarrow_A SH(\mathbb{Q} \cup A)\).

Let \(\bar{a} = \bar{t}(i_0, \ldots, i_n, A_0)\), where \(\bar{t}\) is a sequence of terms of \(\tau^*\), \(i_0 < \ldots < i_n \in I\) and \(A_0 \subseteq A\). We can define a tidy sequence \((j^m_0, \ldots, j^m_n)_{m<\omega}\), \(j^m_0 < \ldots < j^m_n \in I\), such that

1. \(j^m_k = i_k\) for each \(0 \leq k \leq n\),
2. \(j^m_{k+1} = i_k\) is constant if and only if \(i_k \in \mathbb{Q}\),
3. When \(k\) is minimal such that \(i_k \notin \mathbb{Q}\), the indexes at \(k, \ldots, n\) form an \((n - k + 1)\)-block, which is cofinal in \(\kappa + \omega\).

Now the sequence \((\bar{a}_m)_{m<\omega}\), where \(\bar{a}_m = \bar{t}(j^m_0, \ldots, j^m_n, A_0)\), is strongly \(\tau\)-indiscernible by Proposition 3.33(4). Also \(\bar{a}_0 = \bar{a}\). Since for any \(m < \omega\) we have a \(\tau^*\)-isomorphism \(h : SH(\mathbb{Q} \cup A \cup \{j^m_0, \ldots, j^m_n\}) \rightarrow SH(\mathbb{Q} \cup A \cup \{j^m_0, \ldots, j^m_n\})\) fixing \(\mathbb{Q} \cup A\) and mapping \(\bar{a}_m\) to \(\bar{a}\), we get that \(\bar{a}_m \downarrow_A SH(\mathbb{Q} \cup A)\) for each \(m < \omega\).

We claim that \((\bar{a}_m)_{m<\omega}\) is the sequence we need for the proof. To prove the claim, let \(\bar{b}\) be any tuple. Again it is enough to study any \(\bar{b}'\) realizing \(tp^w(\bar{b}/A \cup (\bar{a}_m)_{m<\omega})\), and since \(EM(I, A)\) is \(\aleph_1\)-saturated, we may assume that \(\bar{b} \in EM(I, A)\). Now \(\bar{b} = \bar{t}(h_0, \ldots, h_p, A')\) for some sequence \(\bar{t}'\) of terms of \(\tau^*\), \(h_0 < \ldots < h_p \in I\) and \(A' \subseteq A\). We assume the contrary, that \(\bar{b} \not\equiv_{\bar{A}} \bar{a}_m\) for infinitely many \(m < \omega\). But then by (3) we can find \(m < \omega\) such that \(\bar{b} \not\equiv_{\bar{A}} \bar{a}_m\) and \(h_0 < \ldots < h_p < j^m_k\) for all \(k\) such that \(j^m_k \notin \mathbb{Q}\). There is a partial order-preserving \(f : I \rightarrow I\) fixing \(j^m_0, \ldots, j^m_n\) and mapping \(h_k\) into \(\mathbb{Q}\) for each \(0 \leq k \leq p\). By Proposition 3.33(3), this extends to an \(\tau^*\)-isomorphism \(F\) with domain \(SH(\{j^m_0, \ldots, j^m_n, h_0, \ldots, h_p\} \cup A)\), fixing \(\bar{a}_m \cup A\) and mapping \(\bar{b}\) into \(SH(\mathbb{Q} \cup A)\). Furthermore, since \(dom(F)\) and \(rng(F)\) are models, \(F\) extends to an automorphism of \(\mathfrak{M}\). We get by invariance that \(F(\bar{b}) \not\equiv_{\bar{A}} \bar{a}_m\). But on the other hand, since \(F(\bar{b}) \in SH(\mathbb{Q} \cup A)\), \(\bar{a}_m \downarrow_A F(\bar{b})\). This is a contradiction with finite symmetry. \(\Box\)
\textbf{Theorem 3.38.} Assume that a simple finitary \((\mathbb{K}, \preceq_{\mathbb{K}})\) is \(a\)-categorical in \(\kappa \geq \text{H}\) with uncountable cofinality. Then \((\mathbb{K}, \preceq_{\mathbb{K}})\) is superstable.

\textit{Proof.} Since \(\text{cf}(\text{H}) > \omega\) and \(\max\{2^{\aleph_0}, \text{L}(\mathbb{K})\} < \text{H}\), we have that 
\[ (\max\{2^{\aleph_0}, \text{L}(\mathbb{K})\})^{+\omega} < H \leq \kappa. \]
Denote \(\lambda = (\max\{2^{\aleph_0}, \text{L}(\mathbb{K})\})^{+\omega}\). By Theorem 3.34, \((\mathbb{K}, \preceq_{\mathbb{K}})\) is stable in \(\lambda < \lambda^{\omega}\).

Now by Proposition 3.37, for any \(\vec{a}\) and finite \(A\) such that \(tp^a(\vec{a}/A)\) is unbounded, there is a strongly \(A\)-indiscernible sequence \((\vec{a}_i)_{i<\omega}\) such that for any \(\vec{b}\), the set \(\{i < \omega : \vec{b} \not\models A \vec{a}_i\}\) is finite. Also \(\aleph_0^{\aleph_0} \leq \lambda\). Then by Lemma 3.7, \((\mathbb{K}, \preceq_{\mathbb{K}})\) is superstable. \(\Box\)

In the previous theorem it is enough that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is \(a\)-categorical in some \(\kappa > \lambda \geq \max\{\text{L}(\mathbb{K}), 2^{\aleph_0}\}\), where \(\lambda^{\omega} > \lambda\) and \(\text{cf}(\kappa) > \omega\).

We will prove an \(a\)-categoricity transfer result in section 4.2. For this we need that under superstability, the \(a\)-categorical model of size \(\geq \text{L}(\mathbb{K})\) is \(\text{L}(\mathbb{K})^{+}\)-saturated respect to weak Lascar strong types. We will prove a stronger result: the \(a\)-categorical model is \textit{strongly saturated}. We say that \(\mathcal{A}\) is strongly saturated, if all weak Lascar strong types over subsets of size \(< |\mathcal{A}|\) are realized in \(\mathcal{A}\). We also say that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is \textit{strongly stable} in \(\lambda\), if there are at most \(\lambda\) many weak Lascar strong types over a model of size \(\lambda\).

\textbf{Theorem 3.39.} Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a superstable simple finitary AEC and \(\lambda > \text{L}(\mathbb{K})\). There is a strongly saturated model of size \(\lambda\).

\textit{Proof.} In Corollary 3.4 we show that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is strongly stable in each cardinal \(\lambda \geq \text{L}(\mathbb{K})\). Let \(\lambda > \text{L}(\mathbb{K})\) and let \((\mathcal{A}_i)_{i<\lambda}\) be an increasing and continuous chain of models of size \(\lambda\) such that each weak Lascar strong type over \(\mathcal{A}_i\) is realized in \(\mathcal{A}_{i+1}\). We claim that \(\mathcal{A} = \bigcup_{i<\lambda} \mathcal{A}_i\) is strongly saturated. If \(\lambda\) is regular, this is clear. We may assume that \(\lambda\) is a limit.

Let \(\vec{a}\) be a tuple and \(B \subset \mathcal{A}\) such that \(|B| < \lambda\). We want to realize \(\text{Lstp}^w(\vec{a}/B)\) in \(\mathcal{A}\). By local character for models and since \(\lambda\) is a limit ordinal, there is \(\gamma < \lambda\) such that \(\vec{a} \downarrow_{\mathcal{A}_\gamma} \mathcal{A}\). Denote \(\alpha = \gamma + |B|^+ < \lambda\). Similarly for any finite \(\vec{c}\), there is \(i < \alpha\) such that \(\vec{c} \downarrow_{\mathcal{A}_i} \mathcal{A}_\alpha\). Since \(\text{cf}(\alpha) > |B|\), there is \(\beta\) such that \(\gamma \leq \beta < \alpha\) and 
\[ \vec{c} \downarrow_{\mathcal{A}_\beta} \mathcal{A}_\alpha \] for each finite tuple \(\vec{c} \in B\).

Choose \(\vec{b} \in \mathcal{A}_{\alpha+1}\) realizing \(\text{Lstp}^w(\vec{a}/\mathcal{A}_\beta)\). Then \(\vec{c} \downarrow_{\mathcal{A}_\beta} \vec{b}\) for each finite tuple \(\vec{c} \in B\). By symmetry and finite character over models, \(\vec{b} \downarrow_{\mathcal{A}_\beta} B\). Furthermore by stationarity, \(\vec{b}\) is the realization of \(\text{Lstp}^w(\vec{a}/B)\) in \(\mathcal{A}\). \(\Box\)
To justify the notion of a-categoricity, we give an example of a $L_{\omega_1\omega}$-definable class of structures, which is not categorical but is a-categorical in each cardinal $> 2^{\aleph_0}$.

**Example 3.40.** Let $F$ and $E_i$, $i \leq \omega$ be binary relation symbols. Let $T$ be the following set of axioms:

1. Axioms stating that each $E_i$ and $F$ are equivalence relations.
2. $E_0$ divides the structure into two classes, that is
   \[ \exists x \exists y (\neg E_0(x, y) \land \forall z (E_0(x, z) \lor E_0(y, z))). \]
3. The relation $E_{n+1}$ divides all classes of $E_n$ into two, that is for all $n < \omega$,
   \[ \forall x \exists y \exists z (E_n(y, x) \land E_n(z, x) \land \neg E_n(y, z) \land \forall x' (E_n(x', x) \iff (E_{n+1}(x', y) \lor E_{n+1}(x', z)))). \]
4. The relation $E_{\omega}$ is an intersection of the relations $E_n$, $n < \omega$, that is
   \[ \forall x \forall y (E_\omega(x, y) \iff \bigwedge_{n<\omega} E_n(x, y)). \]
5. All equivalence classes of $E_{\omega}$ are of equal size ($F$ defines a one-to-one and onto function between each two classes), that is
   \[ \forall x \forall y \exists ! z (F(x, z) \land E_\omega(y, z)). \]

The previous example is not categorical, since in a model of $T$ it might happen that some intersection of equivalence classes corresponding to branch in $2^\omega$ is empty. Not even the class of $\aleph_0$-saturated structures of this theory is categorical. If each Lascar strong type over the empty set is realized in a model of $T$, no empty intersections can occur. When $\mathcal{M}$ is an $\alpha$-saturated model of $T$ and is of size $\kappa > 2^{\aleph_0}$, then each equivalence class of $E_\omega$ must be of size $\kappa$. Thus every such model is isomorphic.

## 4. Primary models

In this section we assume that $(\mathcal{K}, \preceq_\mathcal{K})$ is a simple, superstable, finitary AEC with the Tarski-Vaught-property.

**Definition 4.1 (a-isolation).** Let $\bar{a}$ be a tuple and $A$ a set. A type $\text{Lstp}^w(\bar{a}/A)$ is a-isolated over finite $E \subset A$ if whenever $\bar{b}$ realizes $\text{Lstp}(\bar{a}/E)$, then $\bar{b} \downarrow_E A$.

The property of being a-isolated is invariant under automorphisms, that is, if $\text{Lstp}^w(\bar{a}/A)$ is a-isolated over $E \subset A$ and $f \in \text{Aut}(\mathcal{M})$, then $\text{Lstp}^w(f(\bar{a})/f(A))$ is a-isolated over $f(E)$.

**Lemma 4.2.** For every tuple $\bar{a}$, set $A$ and finite $B \subset A$ there is $\bar{b}$ and finite $A_0 \subset A$ such that $\text{Lstp}(\bar{b}/B) = \text{Lstp}(\bar{a}/B)$ and $\text{Lstp}^w(\bar{b}/A)$ is a-isolated over $A_0$. 
Proof. Assume that \( \bar{a}, A \) and finite \( B \subset A \) would witness the contrary. We define tuples \( \bar{a}_i \) and finite sets \( A_i \) for \( i < \omega \) to contradict Proposition 3.11. First let \( \bar{a}_0 = \bar{a} \) and \( A_0 = B \). Then assume we have defined \( \bar{a}_n \) and \( A_n \) for \( i \leq n \) such that

1. \( \text{Lstp}(\bar{a}_i/B) = \text{Lstp}(\bar{a}/B) \),
2. sets \( A_i \) are finite and \( A_i \subset A_{i+1} \subset A \),
3. \( \text{Lstp}(\bar{a}_{i+1}/A_i) = \text{Lstp}(\bar{a}_i/A_i) \) and
4. \( \bar{a}_{i+1} \not\in A_i \).

Since we have (1), the type \( \text{Lstp}^w(\bar{a}_n/A) \) can’t be a-isolated over finite \( A_n \subset A \). Thus there is a tuple \( \bar{a}_{n+1} \) such that \( \text{Lstp}(\bar{a}_{n+1}/A_n) = \text{Lstp}(\bar{a}_n/A_n) \) but \( \bar{a}_{n+1} \not\in A_n \).

Furthermore, by finite character of independence, there is finite \( A_{n+1} \subset A \) such that \( \bar{a}_{n+1} \not\in A_{n+1} \). We may assume that \( A_n \subset A_{n+1} \). This construction contradicts Proposition 3.11.

**Definition 4.3** (a-primary). We say that \( \mathcal{A} \) is \( \mathcal{S} \)-primary over a set \( A \) if for some ordinal \( \xi \) there are tuples \( \bar{a}_i \) and finite sets \( A_i \) for \( i < \xi \) such that

1. the weak Lascar strong type \( \text{Lstp}^w(\bar{a}_i/A \cup \bigcup_{j<i} \bar{a}_j) \) is a-isolated over \( A_i \subset A \cup \bigcup_{j<i} \bar{a}_j \) and
2. \( \mathcal{A} = A \cup \bigcup_{i<\xi} \bar{a}_i \) is \( \mathcal{S} \)-saturated.

If in addition \( \mathcal{A} \) is a-saturated, we say that it is a-primary.

We say that \( \mathcal{A} \) is a-constructible over \( A \), if (1) in the previous definition holds. Analogously to the similar result in [7], we can prove the following lemma.

**Lemma 4.4.** For every set \( A \) there is a model \( \mathcal{B} \) of size \( |A| + \aleph_0 \) such that it is \( \mathcal{S} \)-primary over \( A \). Furthermore, if \( \mathcal{B}' \) is an a-saturated model containing \( A \), we can choose such \( \mathcal{B} \) that \( \mathcal{B} \preceq_{\mathcal{S}} \mathcal{B}' \).

Proof. We prove the last claim. Denote \( |A| + \aleph_0 = \lambda \). By induction on \( n < \omega \) we define sets \( B_n \subset \mathcal{B}' \) of size \( \lambda \), tuples \( \bar{a}_i^n \in \mathcal{B}' \) and finite sets \( A_i^n \subset \mathcal{B}' \) for \( i < \lambda \). First let \( B_0 = A \).

Assume we have defined \( B_n \). Enumerate all finite subsets of \( B_n \) as \( \{(b_j)_{j < \lambda} \} \) and let \( \mathcal{S} = \{\phi_k : k < \omega \} \). Let \( (\bar{c}_j^k)_{j < \lambda, k < \omega} \) be such tuples that whenever there exists a tuple \( \bar{c} \) such that \( \mathcal{M} \models \phi_k(\bar{b}_j, \bar{c}) \) for \( \phi_k \in \mathcal{S} \) and finite \( \bar{b}_j \in B_n \), then one such \( \bar{c} \) is listed as \( \bar{c}_j^k \). If such \( \bar{c} \) does not exist, \( \bar{c}_j^k \) can be arbitrary. Then let \( (\bar{c}_i)_{i < \lambda} \) enumerate all \( (\bar{c}_j^k)_{i < \omega, j < \lambda} \).

Let \( \alpha < \lambda \) and assume we have defined \( \bar{a}_i^n \) for \( i < \alpha \). Let \( \bar{a}_i^n \) be the tuple listed as \( \bar{c}_\alpha \). We use Lemma 4.2 to find a tuple \( \bar{d} \) realizing \( \text{Lstp}(\bar{c}_j^n/b_j) \) and a finite subset \( A_\alpha^n \subset B_n \cup \bigcup_{i<\alpha} \bar{a}_i^n \) such that \( \text{Lstp}^w(\bar{d}/B_n \cup \bigcup_{i<\alpha} \bar{a}_i^n) \) is a-isolated
over $A^n_\alpha$. By a-saturation, there is $\bar{a}_\alpha^n \in \mathcal{B}'$ realizing $\text{Lstp}(\bar{a}/\bar{b}_j \cup A^n_\alpha)$. Then also $\text{Lstp}^w(\bar{a}_\alpha^n/\mathcal{B}_n \cup \bigcup_{i<\alpha} \bar{a}_i^n)$ is $a$-isolated over $A^n_\alpha$. Finally let $B_{n+1} = B_n \cup \bigcup_{i<\alpha} \bar{a}_i^n$.

Clearly $\mathcal{B} = \bigcup_{n<\omega} B_n = A \cup \bigcup_{(n,i) \in \omega \times \lambda} \bar{a}_i^n$ is $S$-saturated and thus a model. Now $\mathcal{B}$ is a $S$-primary model over $A$ and is of size $\lambda$.

We can easily see how to change the previous construction to make an $a$-primary model. In $B_{n+1}$ we should realize also all Lascar strong types over finite subsets of $B_n$. We gain the following result.

**Lemma 4.5.** For every set $A$ there is a model $\mathcal{B}$ of size $|A| + L(\mathcal{K})$ such that it is $a$-primary over $A$. Furthermore, if $\mathcal{B}'$ is an $a$-saturated model containing $A$, we can choose such $\mathcal{B}$ that $\mathcal{B} \preceq A \mathcal{B}'$.

We define domination as usual.

**Definition 4.6** (Domination). We say that a set $A$ dominates a set $B$ over an $a$-saturated model $\mathcal{A}$, if for every tuple $\bar{c}$,

$$\bar{c} \downarrow_{\mathcal{A}} A \Rightarrow \bar{c} \downarrow_{\mathcal{A}} B.$$ 

We show that $a$-primary models have similar properties as $f$-primary models in [7]. Since the concept itself is here different, we need to reprove some of these.

**Lemma 4.7.** Let $B$ be a set and $A_1, A_2 \subseteq B$ finite such that

1. $\text{Lstp}^w(\bar{a}_0/B)$ is $a$-isolated over $A_0$ and
2. $\text{Lstp}^w(\bar{a}_1/B \cup \bar{a}_0)$ $a$-isolated over $A_1 \cup \bar{a}_0$.

Then the type $\text{Lstp}^w(\bar{a}_0, \bar{a}_1/B)$ is $a$-isolated over $A = A_1 \cup A_2$.

**Proof.** We assume the contrary, that there is some $\bar{c}_0 \bar{c}_1$ realizing $\text{Lstp}(\bar{a}_0, \bar{a}_1/A)$ such that $\bar{c}_0 \bar{c}_1 \not\downarrow_A B$. By finite character there is some $\bar{b} \in B$ such that

$$\bar{c}_0 \bar{c}_1 \not\downarrow_A \bar{b}.$$ 

There is $f \in \text{Saut}(\mathfrak{M}/A)$ such that $f(\bar{c}_0 \bar{c}_1) = \bar{a}_0 \bar{a}_1$. By 1, $\bar{c}_0 \downarrow_A \bar{b}$ and then by invariance, $\bar{a}_0 \downarrow_A f(\bar{b})$. Since also $\bar{a}_0 \downarrow_A \bar{b}$, we get from symmetry and stationarity that

$$\text{Lstp}(\bar{b}/A \cup \bar{a}_0) = \text{Lstp}(f(\bar{b})/A \cup \bar{a}_0).$$ 

Let $g$ be a strong automorphism mapping $f(\bar{b})$ to $\bar{b}$ and fixing $A \cup \bar{a}_0$. Now $\text{Lstp}(g(\bar{a}_1)/A \cup \bar{a}_0) = \text{Lstp}(\bar{a}_1/A \cup \bar{a}_0)$ and since by (2),

$$g(\bar{a}_1) \downarrow_{A \cup \bar{a}_0} \bar{b}.$$ 

Pairs Lemma implies that $g(\bar{a}_1) \bar{a}_0 \downarrow_A \bar{b}$. Since $g^{-1} \in \text{Aut}(\mathfrak{M}/\bar{a}_0 \cup A)$ maps $\bar{b}$ to $f(\bar{b})$, $\bar{a}_1 \bar{a}_0 \downarrow_A f(\bar{b})$ by invariance. Again using the automorphism $f^{-1}$ and invariance, we get that $\bar{c}_0 \bar{c}_1 \downarrow_A \bar{b}$, a contradiction. \qed
Proposition 4.8. Let $\mathcal{A}$ be an a-saturated model and $B$ a set. Let $\mathcal{A}^* = \mathcal{A} \cup B \cup \bigcup_{i < \xi} \bar{a}_i$ be a-constructible over $\mathcal{A} \cup B$ and let $\bar{d}$ be a tuple in $\mathcal{A}^*$. Then there are $\bar{a} = \bar{a}_{i_0}, \ldots, \bar{a}_{i_n}$ for $i_0 < \ldots < i_n < \xi$, finite $A \subset \mathcal{A}$ and $\bar{b} \subset B$ such that

1. $\bar{d} \subset A \cup \bar{b} \cup \bar{a}$,
2. $\text{Lstp}^w(\bar{a}/\mathcal{A} \cup \bar{b})$ is a-isolated over $A \cup \bar{b}$ and
3. the tuple $\bar{b}$ dominates $\bar{a} \cup \bar{b}$ over $\mathcal{A}$.

Proof. The proof of items (1) and (2) is identical to the proof of the analogous result in [7], using Lemma 4.7. We assume that we have found $\bar{a}, \bar{b}$ and $A$ satisfying (1) and (2) and then show that (3) holds.

Assume to the contrary, that $\bar{c} \downarrow_A \bar{b}$ but $\bar{c} \not\downarrow_A \bar{b} \cup \bar{a}$ for some tuple $\bar{c}$. By symmetry, also $\bar{b} \not\downarrow_A \bar{c}$. Let $A' \subset \mathcal{A}$ be finite such that $A \subset A'$, $\bar{c} \downarrow_{A'} \mathcal{A} \cup \bar{b}$ and $\bar{b} \downarrow_{A'} \mathcal{A} \cup \bar{c}$. By finite character, there is finite $B'$ such that $A' \subset B' \subset \mathcal{A}$ and $\bar{c} \not\downarrow_{A'} B' \cup \bar{a} \cup \bar{c}$.

Since $\mathcal{A}$ is a-saturated, there is $\bar{d} \in \mathcal{A}$ realizing $\text{Lstp}(\bar{c}/B')$. Since $\bar{b} \downarrow_{B'} \mathcal{A} \cup \bar{c}$, we get $\bar{d} \downarrow_{B'} \bar{b}$ and $\bar{d} \downarrow_{B'} \bar{b}$ by symmetry. By stationarity there is an automorphism $g \in \text{Aut}(\mathcal{M}/B' \cup \bar{b})$ mapping $\bar{c}$ to $\bar{d}$.

By a-isolation we get that $g(\bar{a}) \downarrow_{A' \cup \bar{b}} \mathcal{A}$, and furthermore $g(\bar{a}) \downarrow_{A' \cup \bar{b}} \bar{d} \cup B'$. On the other hand, by monotonicity and invariance $\bar{d} \downarrow_{A'} B' \cup \bar{b}$, and by symmetry, $\bar{b} \cup B' \downarrow_{A'} \bar{d}$. Now by the Pairs Lemma, $g(\bar{a}) \cup \bar{b} \cup B' \downarrow_{A'} \bar{d}$.

But since $\bar{c} \not\downarrow_{A'} \bar{a} \cup \bar{b} \cup B'$, by invariance $\bar{d} \not\downarrow_{A'} g(\bar{a}) \cup \bar{b} \cup B'$, and by symmetry, $g(\bar{a}) \cup \bar{b} \cup B' \not\downarrow_{A'} \bar{d}$, a contradiction. \square

By finite character we gain the following corollary.

Corollary 4.9. Let $\mathcal{B} = \mathcal{A} \cup B \cup (\bar{a}_i)_{i < \xi}$ be an a-constructible model over $\mathcal{A} \cup B$, where $\mathcal{A}$ is an a-saturated model. Then $B$ dominates $\mathcal{B}$ over the model $\mathcal{A}$.

4.1. Morley sequences. In this section we assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple, superstable finitary AEC. The following result is again adapted from Shelah for this context. We sketch the proof.

Proposition 4.10. Let $\lambda$ be a cardinal, $\mathcal{A}$ an a-saturated model, $|\mathcal{A}| \geq \lambda^+ > L(\mathbb{K})$ and let $B \subset \mathcal{A}$ such that $|B| < \lambda^+$. There is an a-saturated model $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}$, finite $E \subset \mathcal{B}$ and a sequence $(\bar{a}_i)_{i < \lambda^+}$ such that $\text{Lstp}^w(\bar{a}_i/\mathcal{B}) = \text{Lstp}^w(\bar{a}_0/\mathcal{B})$ and

$$\bar{a}_i \downarrow_{E \cup \bigcup_{j < i} \bar{a}_j} \text{ for each } i < \lambda^+.$$

Proof. Define a continuous and increasing chain of a-saturated models $\mathcal{A}_i \preceq_{\mathbb{K}} \mathcal{A}$ and tuples $\bar{a}_i \in \mathcal{A}$, $i < \lambda^+$ such that $B \subset \mathcal{A}_0$, $|\mathcal{A}_i| = |B| + L(\mathbb{K})$ and $\bar{a}_i \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$. By superstability, for each $i$ there is finite $E_i \subset \mathcal{A}_i$ such that $\bar{a}_i \downarrow_{E_i} \mathcal{A}_i$. By Fodors
Lemma we may assume that $\lambda^+$-many $E_i$ are included in $\mathcal{A}_0$ for a fixed $i_0$. Taking a subsequence, we may assume that $i_0 = 0$ and furthermore, using the pigeon-hole principle, we may assume that $\bar{a}_i \downarrow_E \mathcal{A}_0$ for a fixed finite $E \subset \mathcal{A}_0$ and for each $i < \lambda^+$. Also since $\lambda^+ > L(\mathbb{K})$, we may assume that $\Lstp(\bar{a}_i/E) = \Lstp(\bar{a}_j/E)$ for each $i < j < \lambda^+$. Then by stationarity, $\Lstp^w(\bar{a}_i/\mathcal{A}_i) = \Lstp^w(\bar{a}_j/\mathcal{A}_j)$ for each $i < j < \lambda^+$. We can take $\mathcal{B} = \mathcal{A}_0$.

We call the sequence $(\bar{a}_i)_{i<\lambda^+}$ from the previous Proposition a Morley-sequence over $\mathcal{B}$. the finite set $E \subset \mathcal{B}$ is called the base set.

Lemma 4.11. Let $(\bar{a}_i)_{i<\alpha}$ be a Morley-sequence over $\alpha$-saturated $\mathcal{B}$, and let $E \subset \mathcal{B}$ be the base set. Then for all $n < \omega$ and $j_0 < ... < j_n < \alpha$,

1. $\bar{b}_{j_0}, ..., \bar{b}_{j_n} \downarrow_E \mathcal{B}$ and
2. $\Lstp^w(\bar{b}_{j_0}, ..., \bar{b}_{j_n}/\mathcal{B}) = \Lstp^w(\bar{b}_0, ..., \bar{b}_n/\mathcal{B})$.

Proof. Item (1) can be showed by induction on $n$, using Pairs Lemma. We prove also (2) by induction on $n$. The case $n = 0$ is clear by definition. We assume that (2) holds for $n$. To prove it for $n + 1$, let $j_0 < ... < j_n < j_{n+1} < \alpha$. Let also $C \subset \mathcal{B}$ be an arbitrary finite subset. By induction, there is $f \in \Saut(\bar{M}/C \cup E)$ such that $f(\bar{b}_{j_k}) = \bar{b}_k$ for $0 \leq k \leq n$. By invariance,

$$f(\bar{b}_{j_{n+1}}) \downarrow_E C \cup \bar{b}_0, ..., \bar{b}_n,$$

and then by stationarity,

$$\Lstp(f(\bar{b}_{j_{n+1}})/E \cup C \cup \bar{b}_n, ..., \bar{b}_0) = \Lstp(\bar{b}_{n+1}/E \cup C \cup \bar{b}_n, ..., \bar{b}_0).$$

Thus

$$\Lstp(\bar{b}_{j_{n+1}}, ..., \bar{b}_{j_0}/C) = \Lstp(f(\bar{b}_{j_{n+1}}), \bar{b}_n, ..., \bar{b}_0/C) = \Lstp(\bar{b}_{n+1}, \bar{b}_n, ..., \bar{b}_0/C).$$

□

4.2. a-categoricity transfer. We use a-primary models to prove an a-categoricity transfer theorem for simple tame finitary AEC. Grossberg and VanDieren have already a categoricity transfer result in [3] for tame classes which can be applied here. The class of a-saturated models of a simple tame finitary AEC with the induced notion $\preceq_K$ forms an abstract elementary class with amalgamation, joint embedding, arbitrarily large models, Löwenheim-Skolem number $L(\mathbb{K})$ and tameness in $L(\mathbb{K})$. Thus the result of [3] implies that a-categoricity in a successor cardinal strictly greater than $L(\mathbb{K})$ gives upwards a-categoricity transfer. Furthermore, then we get a-categoricity for all cardinals above $H(L(\mathbb{K}))$ by the downward categoricity transfer result presented by Baldwin in [1]. Combining these results we get the following theorem.
Theorem 4.12. (Baldwin, Grossberg, Shelah, VanDieren) Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be an AEC with amalgamation, joint embedding, arbitrarily large models and tameness in \(\chi\). Assume that \((\mathbb{K}, \preceq_{\mathbb{K}})\) is a-categorical in a successor cardinal \(\kappa^+ > \max\{L(\mathbb{K})^+, \chi\}\). Then it is a-categorical in every \(\lambda \geq \min\{\kappa^+, H(L(\mathbb{K}))\}\).

Here \(H(L(\mathbb{K}))\) is the Hanf number for the class \((\mathbb{K}_\alpha, \preceq_{\mathbb{K}})\), i.e. the \(\beth\)-model. Our result does not assume the a-categoricity cardinal being a successor, but we still have to make some assumptions on the cardinal. Also the class studied in [3] is more general.

The following proposition is a analogue to the weak categoricity transfer of [7]. Tameness is not needed for this proposition.

Proposition 4.13. Let \((\mathbb{K}, \preceq_{\mathbb{K}})\) be a simple, superstable finitary AEC with the Tarski-Vaught property. Assume that there is \(\kappa > L(\mathbb{K})\) such that each a-saturated model of size \(\kappa\) realizes all weak Lascar strong types over subsets of size \(\leq L(\mathbb{K})\). Then any a-saturated model \(\mathcal{A}\), such that \(|\mathcal{A}| > L(\mathbb{K})\), is saturated respect to weak Lascar strong types.

Proof. Let \(\mathcal{A}\) be an a-saturated model such that \(|\mathcal{A}| > L(\mathbb{K})\) and let \(B \subset \mathcal{A}\), \(|B| < |\mathcal{A}|\). Let also \(\bar{d} \in \mathfrak{M}\) be a finite tuple. By Proposition 4.10 there is a Morley-sequence \((\bar{b}_i)_{i<\omega} \subset \mathcal{A}\) over an a-saturated model \(\mathcal{B} \preceq_{\mathbb{K}} \mathcal{A}\) containing \(B\). Let \(E_1\) be the base set. By local character, there is finite \(E_2\) such that \(\bar{d} \downarrow_{E_2} \mathcal{B}\). We show that \(\operatorname{Lstp}^w(\bar{d}/\mathcal{B})\) is realized in \(\mathcal{A}\).

Let \(\mathcal{C} \preceq_{\mathbb{K}} \mathcal{B}\) be a-saturated and of size \(L(\mathbb{K})\) such that \(E_1 \cup E_2 \subset \mathcal{C}\). We use extension to continue the Morley-sequence to \((\bar{b}_i)_{i<\kappa}\). Let

\[
\mathcal{C}^* = \mathcal{C} \cup \bigcup_{i<\kappa} \bigcup_{j<\xi} \bar{a}_j
\]

be a-primary over \(\mathcal{C} \cup \bigcup_{i<\kappa} \bar{b}_i\) such that \(|\mathcal{C}^*| = \kappa\). By assumption, the model \(\mathcal{C}^*\) is \(L(\mathbb{K})^+\)-saturated respect to weak Lascar strong types. Let \(\bar{d}^* \in \mathcal{C}^*\) realize \(\operatorname{Lstp}^w(\bar{d}/\mathcal{C})\). We find finite \(A \subset \mathcal{C}\), \(\bar{b} = (\bar{b}_0, ..., \bar{b}_m)\), \(i_0 < ... < i_m < \kappa\) and \(\bar{a} = (\bar{a}_{j_0}, ..., \bar{a}_{j_n})\), \(j_0 < ... < j_n < \xi\) as in Proposition 4.8. We use again local character to find finite \(E_3 \subset \mathcal{C}\) such that \(\bar{a} \downarrow_{E_3} \mathcal{C}\).

Denote \(\bar{b}^* = (\bar{b}_0, ..., \bar{b}_m) \in \mathcal{A}\). By Lemma 4.11(2), we have that \(\operatorname{Lstp}^w(\bar{b}/\mathcal{C}) = \operatorname{Lstp}^w(\bar{b}^* / \mathcal{C})\), and thus there is \(f \in \operatorname{Saut}(\mathfrak{M}/E_1 \cup E_3)\) such that \(f(\bar{b}) = \bar{b}^*\). By Corollary 2.12 and simplicity, there is \(\bar{a}'\) realizing \(\operatorname{Lstp}(f(\bar{a})/E_1 \cup E_3 \cup \bar{b}^*)\) such that \(\bar{a}' \downarrow_{E_1 \cup E_3} \mathcal{C}\). By Lemma 4.11(1), \(\bar{b}^* \downarrow_{E_1} \mathcal{C}\), and thus by Pairs Lemma, \(\bar{a}' \downarrow_{E_1} \mathcal{C}\). But now by stationarity,

\[
\operatorname{Lstp}^w(\bar{a}' \upharpoonright \bar{b}^*/\mathcal{C}) = \operatorname{Lstp}^w(\bar{a} \upharpoonright \bar{b}/\mathcal{C}).
\]
Since $\mathcal{A}$ is $a$-saturated, there is $a^* \in \mathcal{A}$ realizing $L_{stp}(a'/A \cup \bar{b}^*)$. By invariance, $L_{stp}(a'/C \cup \bar{b}^*)$ is $f$-isolated over $A \cup \bar{b}^*$. We gain that $a^* \downarrow_{A \cup \bar{b}^*} C$. By stationarity, $a^*$ realizes $L_{stp}(a'/C \cup \bar{b}^*)$. Furthermore by (4.2),

$$(4.3) \quad L_{stp}(a^* \bar{b}^* / C) = L_{stp}(\bar{a} \bar{b}^* / C).$$

By 4.8(3), $\bar{b}^*$ dominates $\bar{b}^* \cup a^*$ over $C$. Since $\bar{b}^* \downarrow_{E_1} B$ by Lemma 4.11(1), using monotonicity, symmetry and dominance we get that

$$(4.4) \quad \bar{a}^* \cup \bar{b}^* \downarrow \in \mathcal{B}.$$ 

But now $L_{stp}(\bar{d}/C)$ is realized by $\bar{d} \subset A \cup \bar{a} \cup \bar{b}$, and $\bar{d} \downarrow \mathcal{B}$ by monotonicity. By (4.4) and stationarity, $L_{stp}(\bar{d}/B)$ is realized in $A \cup \bar{a}^* \cup \bar{b}^* \in \mathcal{A}$. □

We recall the following well-known fact about Galois types.

**Lemma 4.14. (Shelah)** Let $(\mathcal{K}, \preceq_{\mathcal{K}})$ be an AEC with amalgamation, joint embedding and arbitrarily large models.

1. Let $\mathcal{A} \preceq_{\mathcal{K}} \mathcal{B}$, $\mathcal{A} \preceq_{\mathcal{K}} \mathcal{B}'$, $|\mathcal{A}| < |\mathcal{B}'| \leq \kappa$ and $\mathcal{B}$ be $\kappa$-saturated. Then there is an automorphism $f \in \text{Aut}(\mathcal{M}/\mathcal{A})$ such that $f(\mathcal{B}') \preceq_{\mathcal{K}} \mathcal{B}$.

2. Two saturated models $\mathcal{B}_1, \mathcal{B}_2$ containing $\mathcal{A}$, such that $|\mathcal{A}| < |\mathcal{B}_1| = |\mathcal{B}_2|$, are isomorphic over $\mathcal{A}$.

We can embed any countable model to an $\aleph_0$-saturated model. The previous lemma implies that when two models are saturated respect to Galois types, they are isomorphic.

**Theorem 4.15.** Let $(\mathcal{K}, \preceq_{\mathcal{K}})$ be a simple, tame finitary AEC with the Tarski-Vaught property. Assume that $(\mathcal{K}, \preceq_{\mathcal{K}})$ is $a$-categorical in some $\kappa > \lambda \geq \max\{L(\mathcal{K}), 2^{\aleph_0}\}$, where $\lambda^{\aleph_0} > \lambda$ and $cf(\kappa) > \omega$. Then it is $a$-categorical in any $\kappa > L(\mathcal{K})$.

**Proof.** By Theorem 3.34, $(\mathcal{K}, \preceq_{\mathcal{K}})$ is stable in $\lambda$. Then by Proposition 3.37 and Lemma 3.7, the class $(\mathcal{K}, \preceq_{\mathcal{K}})$ is superstable. By Proposition 3.39, the only $a$-saturated model of size $\kappa$ is strongly saturated. Furthermore by Proposition 4.13, then any $a$-saturated model of size $\kappa$ is saturated respect to weak Lascar strong types. Furthermore, by tameness and Theorem 3.20, any $a$-saturated model of size $\kappa$ is saturated respect to Galois types. Then any two $a$-saturated models of the same size $\kappa$ are isomorphic. □

Arguing as in Theorem 3.38, we can show the following corollary. The result is analogous to the Categoricity Conjecture by Shelah, except for the flaw that we assume the $a$-categoricity cardinal to have uncountable cofinality.

**Corollary 4.16.** Assume that $(\mathcal{K}, \preceq_{\mathcal{K}})$ is a simple, tame finitary AEC with the Tarski-Vaught property. If $(\mathcal{K}, \preceq_{\mathcal{K}})$ is $a$-categorical in some $\kappa \geq \lambda$ with uncountable cofinality, it is $a$-categorical in any $\kappa \geq \lambda$. 

4.3. Questions. The first question is motivated by the fact that we would like to drop the assumption of uncountable cofinality in Corollary 4.16. We need this assumption in Proposition 3.37 to ensure that the $\alpha$-categorical model realizes all weak types over countable subsets.

**Question 4.17.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary class, $\alpha$-categorical in some $\kappa > L(\mathbb{K})$ (or $\kappa \geq H$) with countable cofinality. Does the unique $\alpha$-saturated model of size $\kappa$ realize all weak types over countable subsets?

We believe that this question is related to the next question.

**Question 4.18.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple finitary class, $\alpha$-categorical in some $\kappa > L(\mathbb{K})$ with countable cofinality. Is $(\mathbb{K}, \preceq_{\mathbb{K}})$ weakly stable in $\kappa$?

Maybe an even more interesting question is related to the number of Lascar strong types over a finite set in a finitary AEC. In [8] we studied an $\aleph_0$-stable finitary AEC with an additional assumption that the notion splitting would have an extension property. With these assumptions we get that the number of Lascar strong types over any countable set is countable. We do not know how to show this without the extension property for splitting. If one could prove that in a simple $\aleph_0$-stable finitary AEC there are only countably many Lascar strong types over any finite set, also the extension property would follow. This would improve many results of [8]. So we state the question.

**Question 4.19.** Assume that $(\mathbb{K}, \preceq_{\mathbb{K}})$ is a simple $\aleph_0$-stable finitary AEC. Is the number of Lascar strong types over arbitrary finite set always countable?

To gain a negative answer for the previous question we should have a counterexample which is $\aleph_0$-stable but not $\aleph_0$-tame, since $\aleph_0$-tameness implies the extension property for splitting. The counterexample should not be categorical in any $\kappa$ with $\text{cf}(\kappa) > \omega$ either, since in [7](Proposition 3.22) we showed that this implies the number of Lascar strong types over a countable set being countable. We proved in Corollary 3.28 that simplicity and $\aleph_0$-stability imply superstability. This proof did not use tameness nor the extension property for splitting.

One motivation for the study of finitary classes is to generalize the theory of (simple) excellent classes. We have also adapted many methods and concepts from excellent classes, see [9]. Since excellent classes are usually assumed to be $\aleph_0$-stable, this paper can be thought as an attempt to generalize the study of excellent classes beyond $\aleph_0$-stability. We study the superstable case, but one could as well try to study the theory assuming only weak stability. Especially, can we prove stability hierarchy theorem for weak types? Some preliminary results for the behaviour of independence in this case have been studied in section 2.2.
References