Dynamical Systems and Chaos
Homework set 2 (delivery date: 14-February-2006)

Exercise 1
Consider an ODE
\[ \dot{x} = Ax, \]
where
\[ A = \begin{bmatrix} a & 1 \\ 2a & 2 \end{bmatrix}, \]
and \( x : \mathbb{R} \to \mathbb{R}^2 \).

(i) For which value \( a = a^* \in \mathbb{R} \) the qualitative behaviour of the dynamics of the system changes. With the qualitative change of dynamics (bifurcations) we refer to the emergence/disappearance of a fixed point or periodic orbit, a change of stability of a fixed point, etc.

(ii) Describe the phase portrait for values of \( a \) above and below \( a^* \).

Exercise 2
Find general solutions of the following one-dimensional oscillatory equations

(i) \( \ddot{x} + \dot{x} + x = 0 \),

(ii) \( \ddot{x} + 2\dot{x} + x = 0 \).

Exercise 3
Determine a computable condition for a general matrix \( A \in \mathbb{R}^{2 \times 2} \) in
\[ \dot{x} = Ax, \]
that guarantees that the solution \( x : \mathbb{R} \to \mathbb{R}^2 \) travels around the origin in counterclockwise manner.

Exercise 4
Find the general solution \( x : \mathbb{R} \to \mathbb{R}^2 \) and describe the phase portrait for
\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x. \]

Exercise 5
Consider the non-linear system
\[
\begin{align*}
\dot{x}^1 &= |x^2| \\
\dot{x}^2 &= -x^1
\end{align*}
\]
where \( x(t) = [x^1(t) \ x^2(t)]^T \in \mathbb{R}^2 \). Use linear analysis to describe the phase portrait.