Dynamical Systems and Chaos

Homework set 9 (delivery date: 6-April-2006)

Note that the 9th exercise session will be held on Thursday 14-16 in room C123 (the usual space-time coordinates of the lectures) instead of its normal time on Tuesday.

Exercise 1

Read the proof of Floquet’s theorem (notes: Theorem 2.2.3 on p. 35). Let $U(T) = \sum_{j=1}^{m} (\lambda_j P_j + N_j)$, where the matrices $P_i, N_i$ satify their general (defining) commutation relations (See notes: Eq. 2.2.15 on p. 20).

(i) Let $B := \frac{1}{T} \log U(T)$. Give a plausible derivation (don’t worry about convergence issues) of the presentation

$$B = \frac{1}{T} \sum_{j=1}^{m} \left[ \log(\lambda_j) P_j - \sum_{k=1}^{m_j} \frac{1}{j} (-N_j / \lambda_j)^k \right].$$

(ii) Proof (1) by exponentiating the right side.

Exercise 2

Consider a system with normal form

$$\begin{align*}
\dot{x} &= x \left( \lambda + \sum_{j=1}^{N-1} a_j y^j \right) + \mathcal{O}(N + 1) \\
\dot{y} &= \sum_{j=2}^{N} b_j y^j + \mathcal{O}(N + 1)
\end{align*}$$

where $N \geq 2$ and $b_2 \neq 0$.

(i) Infer that on the centre manifold

$$\dot{x} = \mathcal{O}(|y|^{N+2}).$$

(ii) Sketch the flow restricted to the centre manifold when $b$ is positive and negative.

(iii) Draw a local phase portrait at the origin when $\lambda$ and $b$ are both greater than zero.

Exercise 3

Consider the Hill equation

$$\begin{align*}
\ddot{x} &= -\omega^2(t)x \\
\omega(t) &= \begin{cases} \Omega & \text{when } t \in [nT, (n + \frac{1}{2})T), \\ 1 & \text{when } t \in [(n + \frac{1}{2})T, (n + 1)T) \end{cases}, n \in \mathbb{Z},
\end{align*}$$

where $T, \Omega$ are positive parameters. Write the system as a non-autonomous linear first order system. Compute the monodromy matrix and the characteristic exponents. Plot the exponents as a function of $\Omega$ for $T = \pi$. (HINT: Express the eigenvalues of a $2 \times 2$ matrix as a function of its determinant and its trace.)
Exercise 4

Suppose \((t \mapsto \gamma(t))\) is a periodic solution of period \(T\) for a two dimensional system

\[
\dot{x} = f(x)
\]

with \(x(t) \in \mathbb{R}^2\) and \(f \in C^1(\mathbb{R}^2)\). Show that the derivative of the Poincaré map \(\Pi\) along the straight line \(\Sigma\) normal to the loop \(\Gamma = \gamma([0,T])\) at \(x_0 = \gamma(0)\) is given by

\[
\Pi'(0) = \exp \int_0^T \nabla \cdot f(\gamma(t)) \, dt.
\]

Hence the stability of the periodic solution is determined by the sign of this integral.