Dynamical Systems and Chaos
Homework set 10 (delivery date: 11-April-2006)

In this homework set you become familiar with theasic definitions and results of
index theory. Index theory provides global information about the phase
portrait and is related to many deep topological theorems. You should be able to
answer to most of the questions by simply drawing pictures and using geometrical
intuition. This time, no tedious calculations are needed!

Let us begin with an autonomous dynamical system

\[
\begin{bmatrix}
x' \\ y'
\end{bmatrix} = V(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}
\]

on \( \mathbb{R}^2 \). Assume that \( f, g \) are continuously differentiable and the system has unique solutions \( (x(t), y(t)) \) for all times and initial values.

Consider any closed curve \( \Gamma \) in \( \mathbb{R}^2 \) that does not intersect itself and that
does not pass through any fixed point of the system. Assume that \( \Gamma \) is also
continuously differentiable in the sense that there exists a continuously differentiable function \( \gamma : [a, b] \to \mathbb{R}^2 \) with \(-\infty < a < b < \infty \) such that \( \gamma(a) = \gamma(b) \)
and \( \Gamma = \gamma([a, b]) \).

Let us define a function

\[
\theta(x, y) := \arctan \frac{g(x, y)}{f(x, y)},
\]

that measures the angle (modulo \( 2\pi \)) between the vector field \( V \) and the \( x \)-axis
at point \( (x, y) \). If we let a point \( p \in \Gamma \) travel continuously around \( \Gamma \) in counter
clockwise manner the angle \( \theta \) must change continuously. When \( p \) returns to
its starting place, \( \theta \) returns to its original direction. Hence, over one circuit,
\( \theta \) has changed by an integer multiple of \( 2\pi \). Let us denote this integer by a
symbol \( I_V(\Gamma) \). We say that \( I_V(\Gamma) \) is the Poincaré index of the closed curve \( \Gamma \)
with respect to the vector field \( V \).

1. Set \( v(t) := V \circ \gamma(t) \). Derive the expression

\[
I_V(\Gamma) = \frac{1}{2\pi} \int_a^b \frac{v(t) \times \dot{v}(t)}{||v(t)||^2} \, dt,
\]

by differentiating the definition of \( \theta \). Here \( u \times v := u_1v_2 - u_2v_1 \in \mathbb{R} \) for
two vectors \( u, v \in \mathbb{R}^2 \). Notice that the value of the integral is unchanged
if replace \( \gamma \) by another function \( \tilde{\gamma} \) that satisfies the same conditions as \( \gamma \).
(with \( [a, b] \) possibly replaced by another finite interval \( [\tilde{a}, \tilde{b}] \) )

2. Argue that we can deform the loop \( \Gamma \) continuously without changing the
value of the integral (1) as long as the deformed loop does not hit any
critical point \( p^* = (x^*, y^*) \); \( V(p) = 0 \). What can be said about
the behaviour of \( I_V(\Gamma) \) if we instead continuously change the vector field \( V \).

3. By using (2) you should now be able to argue that if the bounded set
(interior set of \( \Gamma \) ) encircled by \( \Gamma \) does not contain fixed points of \( V \) then
\( I_V(\Gamma) = 0 \).
(4) By using the result (3), show that the Poincaré index of the fixed point $p^*$ of $V$ can be defined by setting

$$I_V(p^*) := I_V(\Gamma),$$

where $\Gamma$ is any closed curve whose interior set contains $p^*$ but no other fixed points.

(5) Show that the index $I_V(p^*)$ of a fixed point $p^*$ corresponding to a stable node, unstable node or a center is $+1$. This result implies that the index is not directly related to the stability.

(6) Show that the index of a saddle is $-1$.

(7) Show that

$$I_V(\Gamma) := \sum_{p^* \in \text{int}(\Gamma)} I_V(p^*),$$

where the sum is over all fixed points contained in the interior set $\text{int}(\Gamma)$ of $\Gamma$.

(8) Show that if $\Gamma$ is a closed orbit of the dynamical system, i.e., there is a periodic solution $\dot{p} = V(p)$ of period $T$ such that $\Gamma = p([0, T])$, then the index of $\Gamma$ is $+1$. This implies that a periodic solution of a two dimensional dynamical system must encircle at least one fixed point.