Dynamical Systems and Chaos
Homework set 11 (delivery date: 4-May-2006)

Due the late announcement of this homework set and Wappu the fourth exercise is postponed to 12:th exercise session to be held on 18-May-2006.

Exercise 1
Consider the two dimensional system
\[
\begin{align*}
\dot{x} &= f(x, \mu) \\
\dot{y} &= -y, 
\end{align*}
\] (1)

where \( \mu \in \mathbb{R} \) and \( f(x, \mu) \) is

(i) \( \mu - x^2 \)
(ii) \( \mu x - x^2 \)
(iii) \( \mu x - x^3 \)
(iv) \( \mu x + x^3 - x^5 \)

Sketch phase portraits when \( \mu < 0, \mu = 0, \mu > 0 \), name the bifurcations that occur, and draw bifurcation diagrams \( (\mu, x^*) \) plots for each case (i-iv). What can be said about generality of Equation (1) for higher dimensional systems?

Exercise 2
Consider a two-dimensional system in polar coordinates:
\[
\begin{align*}
\dot{r} &= \mu r + br^3 - \frac{1 + b}{2} r^5 \\
\dot{\theta} &= \omega + c r^3,
\end{align*}
\]

where \( b, c, \mu \in \mathbb{R} \) and \( \omega \neq 0 \). Sketch a phase diagram for \( \mu < 0, \mu = 0, \mu > 0 \) when \( b = -1 \) and \( b = 1 \).

Try to explain why the engineers call the bifurcation of the (radial one-dimensional) system in the case \( b = -1 \) soft or safe, while the bifurcation taking place for the case \( b = +1 \) is called hard or dangerous. Is this terminology motivated for any of the systems in the first exercise?

Exercise 3
Consider the systems
\[
\begin{align*}
(\text{a}) \quad \dot{x} &= \mu + x^2 + y^2, \quad \dot{y} = -y + x^2 \\
(\text{b}) \quad \dot{x} &= \mu + x^2 - y^3, \quad \dot{y} = \mu - y + x^2
\end{align*}
\]

Construct the center manifolds around \((x, \mu, y) = (0, 0, 0)\) for both systems and discuss the dynamics near the origin. What types of bifurcations occur?
Exercise 4 (12th homework set)

Show that the two-dimensional system

\[
\begin{align*}
\dot{x} &= -y + x(x^2 + y^2 - 1)^2 =: f(x, y) \\
\dot{y} &= x + y(x^2 + y^2 - 1)^2 =: g(x, y),
\end{align*}
\]

is not structurally $C^1(K; \mathbb{R}^2)$ stable on any compact set $K \subset \mathbb{R}^2$ which contains the unit disk on its interior. NOTE: The structural stability here refers to stability of the vector field $F(x, y) = [f(x, y) \quad g(x, y)]^T$ in the sense of Definition 2.2.2 ($k = 1$) at p. 24 of the notes Perturbation theory of dynamical systems.