Exercise 10 (Solutions)
5.12.2006

(1) (a) True.
(b) True.
(c) False. For example the cyclic group \( \mathbb{Z}_4 \) and the Klein 4-Group \( V \) are not isomorphic, but both have 4 elements.
(d) True.
(e) False.
(f) True.
(g) True.
(h) False.
(i) True. (Content of Cayley’s Theorem.)

(2) (a) Since \( G \) is cyclic and \( a \in G \) is a generator there exists a \( n \in \mathbb{Z} \) such that \( x = a^n \). Then
\[
f(x) = f(a^n) = f(a^{n-1})f(a) = \ldots = f(a)f(a)^{n-1} = f(a)^n.
\]
Thus the value of \( x \) is determined by the value of \( f(a) \).
(b) Let \( y \in G \) and set \( x := f^{-1}(y) \). Since \( a \in G \) is a generator of \( G \) there exists a \( n \in \mathbb{Z} \) such that \( x = a^n \). Then
\[
y = f(x) = f(a^n) = f(a)^n \in (f(a)).
\]
Therefore \( f(a) \) is a generator of \( G' \). Note that in particular \( G' \) is cyclic.

(3) We know that \( \text{im}(f) \) is a subgroup of \( \mathbb{Z} \). Since \( \mathbb{Z}_n \) is finite it follows that \( \text{im}(f) \) is finite, too. But the only finite subgroup of \( \mathbb{Z} \) is the group \( 0\mathbb{Z} = \{0\} \), thus \( \text{im}(f) = \{0\} \) and it follows that \( f(x) = 0 \) for all \( x \in \mathbb{Z}_n \). Therefore \( f \) is the trivial homomorphism.

(4) • The groups \( \mathbb{Z}_4 \) and \( V \) cannot be isomorphic since \( \mathbb{Z}_4 \) is cyclic and \( V \) is not. The groups \( \mathbb{Z}_4 \) and \( V \) cannot be isomorphic since \( \mathbb{Z}_4 \) has only two elements satisfying the equation \( x^2 = e \) whereas every element in \( V \) satisfies this equation.
• The groups \( S_3 \) and \( \mathbb{Z}_6 \) cannot be isomorphic since \( S_3 \) is not abelian but \( \mathbb{Z}_6 \) is. The groups \( S_3 \) and \( \mathbb{Z}_6 \) cannot be isomorphic since \( S_3 \) is not cyclic but \( \mathbb{Z}_6 \) is. The groups \( S_3 \) and \( \mathbb{Z}_6 \) cannot be isomorphic since \( S_3 \) and \( \mathbb{Z}_6 \) have different lattice diagrams.

(5) See the lecture notes.

(6*) (a) Let \( u, v \in \text{im}(f) \). Then there exists \( x, y \in G \) such that \( u = f(x) \) and \( v = f(y) \). Then
\[
uv = f(x)f(y) = f(xy) = f(y)f(x) = vu
\]
and this shows that \( \text{im}(f) \) is abelian.
(b) If \( f \) is an epimorphism then \( \text{im}(f) = G' \). By (a) this would then mean that \( G' \) is abelian. But this is impossible since \( G' \) is assumed to be not abelian. Therefore \( f \) cannot be an epimorphism.