(1) (a) Not a group. The axiom (G1) is satisfied since the multiplication is associative. The axiom (G2) is satisfied since \(1 \in \mathbb{Z}\) is the unit element. But the axiom (G3) is not satisfied since for example the inverse \(1/2\) of 2 is not an element of \(\mathbb{Z}\).

(b) Not a group. The axiom (G1) is not satisfied since \(a \ast b\) is not associative. For example \((0 \ast 0) \ast 1 = (0 - 0) - 1 = -1 \neq 1 = 0 - (0 - 1) = 0 \ast (0 \ast 1)\).

(c) Is a group. The multiplication is associative and therefore (G1) is satisfied. Further the unit element of the multiplication is 1 and contained in \(\mathbb{R}^+\). Therefore (G2) is satisfied. If \(x > 0\) then also its inverse \(x^{-1} > 0\) and therefore (G3) is satisfied. Altogether is \(\mathbb{R}^+\) a group under multiplication.

(d) Not a group. As before, the multiplication is associative and therefore (G1) is satisfied. The unit element of the multiplication is 1 and contained in \(\mathbb{Q}\). But (G3) is not satisfied since \(0 \in \mathbb{Q}\) but it does not have an inverse.

(e) Is a group. As before (G1) and (G2) is satisfied. Since for every \(x \in \mathbb{R}^*\) holds that \(x \neq 0\) we have that every \(x \in \mathbb{R}^*\) has an inverse. Therefore also (G3) is satisfied.

(f) Is a group. The addition is associative and therefore (G1) is satisfied. The identity element is 0 and is contained in \(\mathbb{R}\) and thus (G2) is satisfied. Finally for every \(x \in \mathbb{R}\) the inverse element is \(-x\) and is contained in \(\mathbb{R}\). Therefore (G3) is also satisfied.

(g) Is a group. The (G1) holds since \((x \ast y) \ast z = x + y + z - 4 = x \ast (y \ast z)\). The element 2 is the identity element since \(x \ast 2 = x + 2 - 2 = x\) and \(2 \ast x = 2 + x - 2 = x\) for every \(x \in \mathbb{Z}\). And if \(x \in \mathbb{Z}\), then \(4 - x \in \mathbb{Z}\) is the inverse element of \(x\). Therefore the axiom (G3) is satisfied.

(2) (a) The major mistakes in this definition are the following two:

- In (ii): “\(x = \) identity” is wrong. The element \(e\) is called the identity.
- In (iii): The symbol of the binary operation is “\(\ast\)” and not “\(\cdot\)”.

In addition to this it would be good to note in the definition of the identity element, that the equation

\[ e \ast x = x \ast e = x \]

holds for every \(x \in G\). Moreover it would be preferable to emphasise in (iii) that \(a'\) is the inverse element of \(a\).

(b) The major mistakes in this definition are the following two:

- A group is a set together with a binary operation. The student mentions the set and the student mentions a operation (and it is a binary operation after all, not just an operation!). But not that both things together make up a group.
- The student mentions the identity element and that every element \(a \in G\) has an inverse element \(a'\). But the student does not give the definition what is meant by “identity element” or what an “inverse element” is. This is missing in the definition.
The major mistakes in this definition are the four following:

- The statement “the binary operation is defined” does not make sense.
- It is not mentioned that the binary operation has to be associative.
- It is logical incorrect to define the inverse element before the identity element!
- Even though the inverse element is mentioned the definition is not complete. One needs to mention that every element \( x \in G \) has an inverse element \( x' \). And one needs to mention what is meant by \( x' \) being the inverse element of \( x \), namely that \( x \cdot x' = x' \cdot x = e \).
- Even though the identity element is mentioned its meaning is not given. One needs to add that for the identity element \( e \) holds that \( x \cdot e = e \cdot x = x \) for every \( x \in G \).

The major mistakes in this definition are the three following:

- In (i): The binary operation \( \ast \) is just associative, not “associative under addition”. The later expression does not make sense.
- In (ii): The expression \( \{ e \} \) is not correct. Correct is just to write “\( e \)”. 
- Moreover in (ii): The effect of the identity element is defined wrong, correct is \( a \ast e = e \ast a = e \) (and not \( a! \)).

(3) (a) Assume that \( y \) is a right inverse of \( x \). Then:

\[ x \ast y = e \]

Multiplying from the left with \( y \):

\[ y \ast x \ast y = y \ast e \]

The element \( e \) being a right unit:

\[ y \ast x \ast y = y \]

Multiplying from the right with the right inverse \( z \) of \( y \):

\[ y \ast x \ast y \ast z = y \ast z \]

Using the fact that \( z \) is the right inverse of \( y \):

\[ y \ast x \ast e = e \]

The element \( e \) being a right unit:

\[ y \ast x = e \]

Therefore \( y \) is a left inverse of \( x \).

(b) As before let \( y \) be a right and therefore also a left inverse of \( x \). Then

\[ x \ast e = x \]

Using \( e = y \ast x \) since \( y \) is also an left inverse of \( x \):

\[ x \ast y \ast x = x \]

Using that \( e = x \ast y \) since \( y \) is an right inverse of \( x \):

\[ e \ast x = x \]

And thus \( e \) is also a left unit.
• We first show that axiom (G2') is satisfied. Fix an element $b \in G$. Then by assumption there exists an $e \in G$ such that

$$b * e = b.$$  

(\ast)

Let $a \in G$ be an arbitrary element. By assumption there exists a $c \in G$ such that

$$c * b = a.$$

Then multiplying the equation (\ast) from the left by $c$ yields

$$c * b * c = c * b$$

and therefore

$$a * e = a.$$

This shows that $e$ is an right unit for every $a \in G$ and thus (G2') is satisfied.

• Next we show that axiom (G3') is satisfied. By assumption there exists for every $x \in G$ a $y \in G$ which satisfies the equation

$$x * y = e.$$

In other words this means that every $x \in G$ has a right inverse $y$. Thus (G3') is satisfied.

(5*) Let $x, y \in G$ and set $z := x * y$. Then

$$x * y = x * e * y$$

$$= x * z * z * y \quad \text{(since } e = z * z)$$

$$= x * x * y * x * y * y \quad \text{(since } z = x * y)$$

$$= e * y * x * e \quad \text{(since } e = x * x \text{ and } e = y * y)$$

$$= y * x$$

Therefore $G$ is an abelian group.