To be discussed on **Tuesday, October 10**. Exercises with a star (*) give extra points.

(1) Find a flaw in the in the following argument: “One does not need the condition (2) in Proposition 2.8 of the lecture notes since it can be derived from (1) and (3) as follows: Let \( x \in H \). Then \( x^{-1} \in H \) by (3) and then \( e = xx^{-1} \in H \) by (1). Therefore (2) is satisfied.”

(2) **Subgroup Criterion, Alternative Version.** Prove Proposition 2.9 of the lecture notes. That is:

(a) Show that if \( H \) is a subgroup of a group \( G \), then \( H \) is a non empty set and \( xy^{-1} \in H \) for every \( x, y \in H \).

(b) Show that if \( H \) is a non-empty subset of \( G \) such that \( xy^{-1} \in H \), then \( H \) is a subgroup of \( G \).

Hint: for part (b) use Proposition 2.8 of the lecture notes. Prove first that \( e \in H \).

(3) Prove Lemma 2.11 of the lecture notes. That is, assume that \( G \) is an abelian group and \( H \) and \( K \) are two subgroups and show that the set

\[ HK := \{ xy : x \in H \text{ and } y \in K \} \]

is a subgroup of \( G \).

(4) Prove the following two results.

(a) Let \( G \) be an abelian group. Show that the set

\[ H := \{ x \in G : x^2 = e \} \]

is a subgroup of \( G \).

(b) Let \( G \) be a group and \( a \in G \) a fixed element. Define the **centralizer** \( C(a) \) of the element \( a \) as

\[ C(a) := \{ x \in G : xa = ax \} \]

Show that the centralizer of \( a \) is a subgroup of \( G \).

Hint: show first that from \( xa = ax \) follows that also \( x^{-1}a = ax^{-1} \).

(c) Let \( G \) be a group. Define the **center** \( Z(G) \) of the group \( G \) as

\[ Z(G) := \{ x \in G : xa = ax \text{ for all } a \in G \} \]

Show that the center of \( G \) is a subgroup of \( G \).

(5*) Let \( G \) be a group and let \( H \) be a subgroup of \( G \). Define the **normalizer** of \( H \) to be

\[ N(H) := \{ x \in G : xHx^{-1} = H \} \]

where \( xHx^{-1} \) denotes the set \( xHx^{-1} := \{ xyx^{-1} : y \in H \} \). Show that \( N(H) \) is a subgroup of \( G \).

Hint: Show first that \( e \in N(H) \). Therefore \( N(H) \) is not empty. Next show that if \( xHx^{-1} = H \) then also \( x^{-1}Hx = H \). Now use Proposition 2.9 of the lecture notes. Remember also Lemma 2.5 of the lecture notes.