(1) (a) False.
   (b) True.
   (c) False. It is desirable to show that no permutation can be expressed as a product of both an even and an odd number of transpositions, before defining even and odd permutations.
   (d) False.
   (e) False. $|A_5| = 5!/2 = 120/2 = 6$.
   (f) False. The group $S_1 = \{\text{id}\}$ and $S_2 = \{\text{id}, \sigma\}$ are cyclic (and they are the only two examples of cyclic symmetric groups).
   (g) True. $|A_3| = 3!/2 = 3$. There exists only one type of group with 3 elements and this group is the cyclic group of 3 elements. Therefore $A_3$ is cyclic.
   (h) True.
   (i) True.
   (j) False. This subset is not closed under the multiplication and neither does it contain the identity element (which is always an even permutation).
   (k) True. This subgroup is the $A_{10}$.

(2) (a) \begin{itemize}
   \item $\sigma_1 = (1\ 4\ 5\ 8\ 7\ 2)$ ($\sigma_1$ is a cycle of length 6).
   \item $\sigma_1 = (1\ 2)(1\ 7)(1\ 8)(1\ 5)(1\ 4)$ ($\sigma_1$ is an odd permutation).
\end{itemize}
   (b) \begin{itemize}
   \item $\sigma_2 = (1\ 3\ 2\ 7)(4\ 8\ 6)$ ($\sigma_2$ is a product of 2 disjoint cycles, one of length 4 and one of length 3).
   \item $\sigma_2 = (1\ 7)(1\ 2)(1\ 3)(4\ 6)(4\ 8)$ ($\sigma_2$ is an odd permutation).
\end{itemize}
   (c) \begin{itemize}
   \item $\sigma_3 = (1\ 5\ 8\ 2\ 4\ 7)$ ($\sigma_3$ is a cycle of length 6).
   \item $\sigma_3 = (1\ 7)(1\ 4)(1\ 2)(1\ 8)(1\ 5)$ ($\sigma_3$ is an odd permutation).
\end{itemize}

(3) (a) We calculate successively

$$
\begin{align*}
\sigma_4 &= (1\ 4\ 5\ 7) \\
\sigma_2^2 &= (1\ 4\ 5\ 7)(1\ 4\ 5\ 7) = (1\ 5)(4\ 7) \\
\sigma_1^2 &= (1\ 5)(4\ 7)(1\ 4\ 5\ 7) = (1\ 7\ 5\ 4) \\
\sigma_1^4 &= (1\ 7\ 5\ 4)(1\ 4\ 5\ 7) = (1)
\end{align*}
$$

and

$$
\begin{align*}
\sigma_5 &= (3\ 4\ 6) \\
\sigma_2^2 &= (3\ 4\ 6)(3\ 4\ 6) = (3\ 6\ 4) \\
\sigma_1^2 &= (3\ 6\ 4)(3\ 4\ 6) = (1)
\end{align*}
$$

Thus $\text{ord}(\sigma_4) = 4$ and $\text{ord}(\sigma_5) = 3$.

(b) Let $\sigma \in S_n$ be a cycle of length $\ell$. Then $\text{ord}(\sigma) = \ell$.

(c) \begin{itemize}
   \item Note that $\sigma_6 = \tau\mu$ with $\tau := (4\ 5)$ and $\mu := (2\ 3\ 7)$. Since $\tau$ and $\mu$ are disjoint cycles it follows that $\tau$ and $\mu$ commute, that is $\tau\mu = \mu\tau$. Therefore we get the simple formula $\sigma_6^k = (\tau\mu)^k = \tau^k\mu^k$.
\end{itemize}
Since \( \tau \) is a transposition we get that \( \tau^2 = \text{id} \). Further we have for the cycle \( \mu: \mu^2 = (2\ 3\ 7)(2\ 3\ 7) = (2\ 7\ 3) \neq \text{id} \) and then \( \mu^3 = (2\ 7\ 3)(2\ 3\ 7) = \text{id} \).

Using this information we calculate then
\[
\begin{align*}
\sigma_0^2 &= \tau^2 \mu^2 = \mu^2 \neq \text{id} \\
\sigma_0^3 &= \tau^3 \mu^3 = \tau \neq \text{id} \\
\sigma_0^4 &= \tau^4 \mu^4 = \mu \neq \text{id} \\
\sigma_0^5 &= \tau^5 \mu^5 = \tau \mu^2 \neq \text{id} \\
\sigma_0^6 &= \tau^6 \mu^6 = \text{id}
\end{align*}
\]
Therefore we get \( \text{ord}(\sigma_0) = 6 \). (Note that 6 is a least common multiple of 2 and 3.)

- Note that \( \sigma_2 = \tau \mu \) with \( \tau := (1\ 4) \) and \( \mu := (3\ 5\ 7\ 8) \). As before we conclude that \( \tau \) and \( \mu \) commute since they are disjoint cycles. Therefore we get the simple formula \( \sigma_2^k = (\tau \mu)^k = \tau^k \mu^k \).

Since \( \tau \) is a transposition we have \( \tau^2 = \text{id} \). Further we have for the cycle \( \mu: \mu^2 = (3\ 5\ 7\ 8)(3\ 5\ 7\ 8) = (3\ 7)(5\ 8) \neq \text{id} \), \( \mu^3 = (3\ 7)(5\ 8)(3\ 5\ 7\ 8) = (3\ 8\ 7\ 5) \neq \text{id} \) and then finally \( \mu^4 = (3\ 8\ 7\ 5)(3\ 5\ 7\ 8) = \text{id} \).

Using this information we calculate then
\[
\begin{align*}
\sigma_2^2 &= \tau^2 \mu^2 = \mu^2 \neq \text{id} \\
\sigma_2^3 &= \tau^3 \mu^3 = \tau \neq \text{id} \\
\sigma_2^4 &= \tau^4 \mu^4 = \mu \neq \text{id}
\end{align*}
\]
Therefore we get \( \text{ord}(\sigma_2) = 4 \). (Note that 4 is a least common multiple of 2 and 4.)

- The permutation \( \sigma_1 \) is a cycle of length 6. This suggests that \( \text{ord}(\sigma_1) = 6 \). Explicite calculation verifies this.

- The permutation \( \sigma_2 \) has a decomposition into a product of two disjoint cycles, one of length 3 and one of length 4. A least common multiple of 3 and 4 is 12 and this suggests that \( \text{ord}(\sigma_2) = 12 \). Explicite calculation verifies this.

- The permutation \( \sigma_3 \) is a cycle of length 6. This suggests that \( \text{ord}(\sigma_3) = 6 \). Explicite calculation verifies this.

Let \( \sigma \in S_n \) and assume that \( \sigma = \mu_1 \cdots \mu_k \) is a decomposition of \( \sigma \) into pairwise disjoint cycles \( \mu_i \) of length \( \ell_i \). Then the order of \( \sigma \) is a least common multiple of the integers \( \ell_1, \ldots, \ell_k \), that is \( \text{ord}(\sigma) \in \text{lcm}(\ell_1, \ldots, \ell_k) \).

\((4*)\) (a) A permutation of the set \( G \) is a bijective map \( G \to G \). Thus we have to show three things:

(i) \( \lambda_a \) is indeed a map \( G \to G \). But this is clearly true, since \( ax \in G \) for every \( x \in G \).

(ii) \( \lambda_a \) is surjective. Therefore let \( y \in G \) be an arbitrary element. Set \( x := a^{-1}y \), which is an element in \( G \), too. By construction we have now that \( \lambda_a(x) = ax = a \cdot a^{-1}y = ey = y \). Thus \( \lambda_a \) is indeed surjective.

(iii) \( \lambda_a \) is injective. Therefore assume that \( x, y \in G \) are elements such that \( \lambda_a(x) = \lambda_a(y) \). This means that \( ax = ay \) and since in a group the left cancelation law holds we get that \( x = y \). Thus \( \lambda_a \) is indeed injective.

Alltogether this shows that \( \lambda_a \) is a permutation of the set \( G \).

(b) For any \( x \in G \) we get
\[
\lambda_{ab}(x) = abx = a(bx) = a \cdot \lambda_b(x) = \lambda_a(\lambda_b(x)) = (\lambda_a \lambda_b)(x).
\]
Therefore \( \lambda_{ab} = \lambda_a \lambda_b \) as elements of \( S_G \).
(c) For every $x \in G$ we have $\lambda_e(x) = ex = x$. Thus $\lambda_e$ is the identity map of $G$ and therefore $\lambda_e$ is the identity element of the group $SG$.

(d) Using the previous results we get

$$\lambda_{a^{-1}} \lambda_a = \lambda_{a^{-1}a} = \lambda_e.$$ 

Thus $\lambda_{a^{-1}}$ is indeed the inverse element of $\lambda_a$. That is $\lambda_{a^{-1}} = (\lambda_a)^{-1}$. 