To be discussed on Tuesday, November 21. Exercises with a star (*) give extra points.

(1) Mark each of the following true or false.
   a) Every cyclic group is abelian.
   b) Every abelian group is cyclic.
   c) \( \mathbb{Q} \) under the addition is a cyclic group.
   d) Every element of a cyclic group group \( G \) generates \( G \).
   e) For every integer \( n > 0 \) there exists at least one abelian group \( G \) with \( \text{ord}(G) = n \).
   f) Every group of order \( \leq 4 \) is cyclic.
   g) \( S_3 \) is a cyclic group.
   h) \( A_3 \) is a cyclic group.
   i) Let \( G \) be a group such that every proper subgroup of \( G \) is cyclic. Then \( G \) is cyclic.

(2) Greatest Common Divisor. Recall that a integer \( p \geq 2 \) is called a prime number if the only positive integers dividing \( p \) are 1 and \( p \) itself. The set of all prime numbers is denoted by \( \mathbb{P} \), that is

\[ \mathbb{P} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots \} \]

which shows the the first 13 prime numbers.

A greatest common divisor \( d \) of a set of non-zero integers \( \{a_1, \ldots, a_k\} \) is a integer which divides every \( a_i \) and such that every integer \( m \) which divides every integer \( a_i \) divides also \( d \).

Given set \( \{a_1, \ldots, a_k\} \) of non-zero integers one can find a greatest common divisor as follows: for every prime number \( p \in \mathbb{P} \) let \( n^p \) be the largest integer such that \( p^n \) divides every \( a_i \) (\( 1 \leq i \leq k \)). Then

\[ d := \prod_{p \in \mathbb{P}} p^{n^p} = 2^{n_2} \cdot 3^{n_3} \cdot 5^{n_5} \cdot 7^{n_7} \ldots \]

is a greatest common divisor of the set \( \{a_1, \ldots, a_k\} \) of non-zero integers.

a) Every integer can be written as a product of prime numbers. Write the following integers as a product of prime numbers:

\[ 15, 21, 24, 27, 28, 35, 36, 30, 40, 42, 46 \text{ and } 48. \]

b) Determine a greatest common divisor for the following sets:

\[ \{15, 40\} \quad \{24, 36\} \quad \{21, 48\} \]

\[ \{21, 41\} \quad \{30, 36\} \quad \{15, 21\} \]

\[ \{21, 46, 23\} \quad \{30, 42, 46\} \quad \{-7, 21, 28\} \]

(3*) Let \( H \) be a cyclic group with \( m \) elements and \( K \) a cyclic group with \( n \) elements. Let \( x \) be a generator of \( H \) and let \( y \) be a generator of \( K \). Then the group

\[ G := H \times K \]

has \( nm \) elements.

a) Show that if \( nm \) is a least common multiple of \( n \) and \( m \), then \( (x, y) \) is a generator of \( G \).
(b) Show that if $nm$ is not a least common multiple of $n$ and $m$, then $(x, y)$ is not a generator of $G$.

(c) Show that if $(x', y')$ is a generator of $G$ then $x'$ is a generator of $H$ and $y'$ is a generator of $K$.

(d) Use the results of part (b) and (c) to conclude that if $nm$ is not a least common multiple of $n$ and $m$, then $G$ is not cyclic.

Note that the above exercise explains why the Klein 4-Group $V = C_2 \times C_2$ is not cyclic, whereas the group $C_2 \times C_3$ is cyclic (see exercises 3.2 and 4.4): the least common multiple of 2 and 2 is again 2 (and not 4) in the case of the Klein 4-Group whereas the least common multiple of 2 and 3 is 6 which is the order of $C_2 \times C_3$. 