(1) This is a straight application of Case 2 and Case 3 of Proposition 1.1 of the lecture notes.

We have

\[
\delta(1, -c, (c - 1), -2) = -2 + c(c - 1) = c^2 - c - 2 = c^2 - 2 \cdot c \cdot \frac{1}{2} + \frac{1}{4} - \frac{9}{4} = \left(c - \frac{1}{2}\right)^2 - \frac{9}{4} = 0
\]

if and only if

\[
c = \frac{1 \pm 3}{2},
\]

that is if and only if \(c = 2\) or \(c = -1\).

- Let us first consider the case that \(c \neq 2\) and \(c \neq -1\). Then the third case of Proposition 1.1 applies which states that the system of linear equations has a unique solution and this solution is given by:

\[
x = \frac{\delta(1, -1, 1, -2)}{\delta(1, -c, (c - 1), -2)} = \frac{c - 2}{c(c - 1) - 2}
\]

\[
y = \frac{\delta(1, 1, (c - 1), 1)}{\delta(1, -c, (c - 1), -2)} = \frac{c + 2}{c(c - 1) - 2}
\]

- Next let us consider the case \(c = 2\). In this case the system of linear equations looks like:

\[
x - 2y = 1
\]

\[
x - 2y = 1
\]

That is, both equations are equal. Clearly we have that

\[
x := 2t + 1 \quad \text{and} \quad y := t
\]

is a solution for every \(t \in \mathbb{R}\). Thus we have more than one solution in the case \(c = 2\).

- Last we consider the case \(c = -1\). In this last case the system of linear equations looks like:

\[
x + y = 1
\]

\[
-2x - 2y = 1
\]

But this system of linear equations can never be satisfied because if \(x, y \in \mathbb{R}\) such that \(x + y = 1\) then the second line reads \(-2 = 1\). Thus in the case \(c = -1\) we get that there is no solution for the given system of linear equations.
(2) (a) Let \( x, x' \in M \) be two magic squares. Assume that the row, column and diagonal sums of \( x \) are all equal to \( c \) and assume that the row, column and diagonal sums of \( x' \) are all equal to \( c' \). Then the row, column and diagonal sums of \( x + x' \) are all equal to \( c + c' \) and thus \( x + x' \) is a magic square, that is \( x + x' \in M \). If \( a \) is a number, then the row, column and diagonal sums of \( ax \) are all equal to \( ac \) and thus \( ax \) is a magic square, too, that is \( ax \in M \). Thus altogether we have shown that \( M \) is a subspace of \( \mathbb{R}^3 \).

If now both \( x, x' \in M_0 \), then \( c = c' = 0 \) and then \( c + c' = 0 \) and \( a \cdot c = 0 \). Therefore \( x + x' \in M_0 \) and \( ax \in M_0 \), too. This shows that \( M_0 \) is also a subspace of \( \mathbb{R}^3 \).

There are many ways to see that \( M_c \) cannot be a subspace of \( \mathbb{R}^3 \) for \( c \neq 0 \). For example if \( x \in M_c \), then the row, column and diagonal sum of \( 2x \) are all equal to \( 2c \neq c \). Therefore \( 2x \notin M_c \) and \( M_c \) is not a subspace of \( \mathbb{R}^3 \) because it is not closed under the scalar multiplication. (Similar one can also see that \( M_c \) is not closed under the addition, but one counter example is already enough.)

(b) Assume that \( x \) is a magic square with row, column and diagonal sum equal to \( c \). Then the coefficients of \( x \) satisfy the following nonhomogeneous system of 8 linear equations in 9 unknown variables

\[
\begin{align*}
x_1 + x_2 + x_3 &= c \\
x_4 + x_5 + x_6 &= c \\
x_7 + x_8 + x_9 &= c \\
x_1 + x_4 + x_7 &= c \\
x_2 + x_5 + x_8 &= c \\
x_3 + x_6 + x_9 &= c \\
x_1 + x_2 + x_3 &= c \\
x_4 + x_5 + x_6 &= c \\
\end{align*}
\]

and any solution \( x \) to this system of linear equations is a magic square. The first three equations of (\(*\)) say that all row sums are equal to \( c \), the next three equations say that all column sums are equal to \( c \) and the last two equations say that all diagonal sums are equal to \( c \).

(c) In order to answer the question why \( M_c \neq \emptyset \) we need to study the solutions of the system of linear equations (\(*\)). Now using elementary transformations we can transform (\(*\)) into the following equivalent one system of linear equations:

\[
\begin{align*}
x_1 + x_9 &= \frac{1}{3}c \\
x_2 + x_8 &= \frac{1}{3}c \\
x_3 - x_8 - x_9 &= \frac{1}{3}c \\
x_4 - x_8 - 2x_9 &= \frac{1}{3}c \\
x_5 &= \frac{2}{3}c \\
x_6 + x_8 + 2x_9 &= \frac{4}{3}c \\
x_7 + x_8 + x_9 &= c
\end{align*}
\]

Now this system of linear equations is clearly solvable for every value of \( c \) (choose arbitrary values for \( x_8 \) and \( x_9 \) and then calculate the values for the remaining variables \( x_1, \ldots, x_7 \) using the constraints given by the equations). Thus \( M_c \neq \emptyset \) for every value of \( c \). Moreover we see from the 3-th equation that for every magic square holds \( x_5 = c/3 \) or equivalently \( 3x_5 = c \).
Now from $3x_5 = c$ follows in the case that $x \in M_0$ that $x_5 = 0$ and therefore

$$x := \begin{pmatrix} a & -a - b & b \\ -a + b & 0 & a - b \\ -b & a + b & -a \end{pmatrix}$$

for some numbers $a$ and $b$. And for any numbers $a$ and $b$ above is a magic square with row, column and diagonal sum equal to 0.

(3) (a) Let $\alpha, \beta, \gamma \in \mathbb{R}$ be any three numbers. We want to construct a $3 \times 3$-matrix $x$ such that $x_1 = \alpha$, $x_2 = \beta$, $x_3 = \gamma$ and such that $x$ is a magic square. That is, we want to complete

$$x = \begin{pmatrix} \alpha & \beta & \gamma \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

in such a way that the row, column and diagonal sums are all equal. So what can we do? First of all, since a first row is given, we can calculate the row sum $c := \alpha + \beta + \gamma$. From the exercises before we know that in a magic square always the equality

$$x_5 = c/3$$

holds. Thus we are forced to set $x_5 := c/3 = (\alpha + \beta + \gamma)/3$ as we want $x$ to become a magic square. Note that we have no other choice for $x_5$. Therefore we get:

$$x = \begin{pmatrix} \alpha & \beta & \gamma \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

But now we see that from condition that the diagonal and column sums need all to be equal to $c$ determines uniquely the values for $x_7$, $x_8$ and $x_9$ since we have that the following three equations need to be satisfied:

$$\alpha + \frac{\alpha + \beta + \gamma}{3} + x_9 = \alpha + \beta + \gamma$$

$$\beta + \frac{\alpha + \beta + \gamma}{3} + x_8 = \alpha + \beta + \gamma$$

$$\gamma + \frac{\alpha + \beta + \gamma}{3} + x_7 = \alpha + \beta + \gamma$$

These equations are satisfied if and only if we choose the values for $x_7$, $x_8$ and $x_9$ as follows:

$$x_7 := \frac{2\alpha + 2\beta - \gamma}{3}$$

$$x_8 := \frac{2\alpha - \beta + 2\gamma}{3}$$

$$x_9 := \frac{-\alpha + 2\beta + 2\gamma}{3}$$

With this choice our becoming magic square looks now as follows:

$$x = \begin{pmatrix} \alpha & \beta & \gamma \\ x_4 & \frac{\alpha + \beta + \gamma}{3} & x_6 \\ \frac{2\alpha + 2\beta - \gamma}{3} & \frac{2\alpha - \beta + 2\gamma}{3} & \frac{-\alpha + 2\beta + 2\gamma}{3} \end{pmatrix}$$

Now the condition that the column sums need to be equal to $c$ we can determine what values we need to choose for the remaining variables
\[ \begin{align*} x_4 \text{ and } x_5 \text{ as these two variables need to satisfy the following two equations: } & \\
\alpha + x_4 + \frac{2\alpha + 2\beta - \gamma}{3} = \alpha + \beta + \gamma & \\
\beta + x_6 + \frac{-\alpha + 2\beta + 2\gamma}{3} = \alpha + \beta + \gamma & \\
\end{align*} \]

These equations are satisfied if and only if we choose for \( x_4 \) and \( x_5 \) the following values:

\[ \begin{align*} x_4 & := \frac{-2\alpha + \beta + 4\gamma}{3} \\
x_6 & := \frac{4\alpha + \beta - 2\gamma}{3} & \\
\end{align*} \]

Thus we have now completed our \( 3 \times 3 \)-matrix to the following form:

\[ x = \begin{pmatrix} \alpha & \beta & \gamma \\
\frac{-2\alpha + \beta + 4\gamma}{3} & \frac{\alpha + \beta + \gamma}{3} & \frac{4\alpha + \beta - 2\gamma}{3} \\
\frac{2\alpha - \beta + 2\gamma}{3} & \frac{2\alpha - \beta + 2\gamma}{3} & \frac{-\alpha + 2\beta + 2\gamma}{3} \end{pmatrix} \]

But is it indeed a magic square?! So far we have chosen the values for \( x_4, \ldots, x_9 \) such that the all column and diagonal sums are equal to the sum of the sum of the first row. But we need still to verify that by our choice also the sum of the second and last row are equal to \( \alpha + \beta + \gamma \).

But this is indeed the case since

\[ \begin{align*} \frac{-2\alpha + \beta + 4\gamma}{3} + \frac{\alpha + \beta + \gamma}{3} + \frac{4\alpha + \beta - 2\gamma}{3} &= \frac{3\alpha + 3\beta + 3\gamma}{3} \\
&= \alpha + \beta + \gamma \end{align*} \]

and

\[ \begin{align*} \frac{2\alpha + 2\beta - \gamma}{3} + \frac{2\alpha - \beta + 2\gamma}{3} + \frac{-\alpha + 2\beta + 2\gamma}{3} &= \frac{3\alpha + 3\beta + 3\gamma}{3} \\
&= \alpha + \beta + \gamma \end{align*} \]

Thus we obtained really a magic square. Note that we had only one possibility to complete the 6 unknown variables!

(b) It is in general not possible to complete \( x \) to a magic square if the first two rows are given. For example if the row sum of the two given rows do not equal each other then it is already impossible. But even if the row sum of both rows equal each other and even if \( x_5 = (x_1 + x_2 + x_3)/3 \) holds it might still be impossible to complete \( x \) to a magic square. This happens if

\[ \begin{align*} x_4 & \neq \frac{-2x_1 + x_2 + 4x_3}{3} \\
x_5 & = \frac{1}{3}(x_1 + x_2 + x_3) \\
x_6 & = \frac{2}{3}(x_1 + x_2 + x_3) - x_4 \\
\end{align*} \]

(4*) (a) The row, column and diagonal sums of \( e_2 \) and \( e_3 \) are all equal to 0 and thus \( e_2, e_3 \in M_0 \). By the second exercise we know that \( M_0 \) is a subspace and therefore \( t_2 e_2 + t_3 e_3 \in M_0 \) for any numbers \( t_2 \) and \( t_3 \).

Now if \( x \in M_0 \) is given and we want to find numbers \( t_2 \) and \( t_3 \) such that

\[ x = t_2 e_2 + t_3 e_3 \]

\[ (\ast \ast \ast) \]
then at least the following three equations must be satisfied:

\[
\begin{align*}
t_2 &= x_1 \\
-t_2 + t_3 &= x_2 \\
-t_3 &= x_3
\end{align*}
\]

which is derived from the fact that the first row of the left hand side of (\(* \ast \ast \ast\)) must agree with the first row of the right hand side of (\(* \ast \ast \ast\)). This is a nonhomogeneous system of 3 linear equations in the two unknown variables \(t_2\) and \(t_3\). Using elementary transformations one can transform this system into the following equivalent system of linear equations:

\[
\begin{align*}
t_2 &= x_1 \\
t_3 &= -x_3 \\
0 &= x_1 + x_2 + x_3
\end{align*}
\]

Since \(x \in M_0\) it follows that \(x_1 + x_2 + x_3 = 0\) and thus the last row of the equation is satisfied. Furthermore there exists only a unique solution, namely \(t_2 := x_1\) together with \(t_3 := -x_3\).

We still have to show that for these values of \(t_2\) and \(t_3\) indeed the equality (\(* \ast \ast \ast\)) holds. Therefore we try to determine 

\[x' := x - t_2e_2 - t_3e_3\]

and show that \(x' = 0\). We know that \(x'\) is a magic square because \(M_0\) is a subspace. Further we know that the first row of \(x\) coincides with the first row of \(t_2e_2 + t_3e_3\) and thus all the coefficients of the first row of \(x'\) are equal to zero. By the first third exercise we know that then \(x'\) can then only be the zero matrix (because there is exactly only one way to complete the first row to a magic square and all the coefficients of this magic square will be equal to zero). Therefore \(x = t_2e_2 + t_3e_3\) for \(t_2 = x_1\) and \(t_3 = -x_3\) and these numbers are uniquely determined by \(x\).

(b) We could use the similar reasoning as in the first part. An alternative reasoning is the following:

Denote the row sum of \(x\) by \(c\). Then there exists precisely one \(t_1\) such that \(x - t_1e_1 \in M_0\), namely \(t_1 = c/3\). Then by the result of the first part we have that there exists unique numbers \(t_2\) and \(t_3\) (namely \(t_2 = x_1 - c/3\) and \(t_3 = -(x_3 - c/3)\)) such that \(x - t_1e_1 = t_2e_2 + t_3e_3\). Therefore we have seen that there exists unique numbers \(t_1\), \(t_2\) and \(t_3\) such that 

\[x = t_1e_1 + t_2e_2 + t_3e_3.\]