(1) (a) Following the example on page 87 in the lecture notes gives:

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
3 & 4 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & -2 & -3 & 1 \\
1 & 0 & -2 & 1 \\
0 & -2 & -3 & 1 \\
1 & 0 & -2 & 1 \\
0 & 1 & \frac{3}{2} & \frac{1}{2}
\end{bmatrix}
\]

Therefore the inverse matrix of

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\]

is the matrix

\[
\begin{bmatrix}
-2 & 1 \\
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
\].

(b) Following the example on page 87 in the lecture notes gives:

\[
\begin{array}{cccc}
-13 & 7 & -16 & 1 & 0 & 0 \\
-28 & 15 & -35 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 1 \\
\hline
-13 & 7 & -16 & 1 & 0 & 0 \\
-2 & 1 & -3 & -2 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 1 \\
\hline
1 & 0 & 8 & 15 & -7 & 0 \\
-2 & 1 & -3 & -2 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 \\
\hline
1 & 0 & 0 & 15 & -7 & 4 \\
-2 & 1 & -3 & -2 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 \\
\hline
1 & 0 & 0 & 15 & -7 & 4 \\
0 & 1 & -3 & 28 & -13 & 8 \\
0 & 0 & 2 & 0 & 0 & -1 \\
\hline
1 & 0 & 0 & 15 & -7 & 4 \\
0 & 1 & 0 & 28 & -13 & \frac{13}{2} \\
0 & 0 & 1 & 0 & 0 & -\frac{1}{2}
\end{array}
\]

Therefore the inverse matrix of

\[
\begin{bmatrix}
-13 & 7 & -16 \\
-28 & 15 & -35 \\
0 & 0 & -2
\end{bmatrix}
\]

is the matrix

\[
\begin{bmatrix}
15 & -7 & 4 \\
28 & -13 & \frac{13}{2} \\
0 & 0 & -\frac{1}{2}
\end{bmatrix}
\].

Note how in the above solution we avoided the use of fractions till the very last moment in order to keep the transformations more simple. This helps avoiding mistakes in the calculation.
(c) Following the example on page 87 in the lecture notes gives:

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 0 \\
2 & 5 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

Therefore the inverse matrix of

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

is the matrix

\[
\begin{bmatrix}
5 & -2 & -1 \\
-2 & 1 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

(2) (a) According to the note on page 75 of the lecture notes the transition matrix \( S \) from the standard basis of \( \mathbb{R}^3 \) to the basis \( B \) is

\[
S := \begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

From the previous exercise we know that its inverse is

\[
S^{-1} = \begin{bmatrix}
5 & -2 & -1 \\
-2 & 1 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

By Theorem 3.42 we have \( A' = S^{-1}AS \). That is

\[
A' = \begin{bmatrix}
5 & -2 & -1 \\
-2 & 1 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
-13 & 7 & -16 \\
-28 & 15 & -35 \\
0 & 0 & -2 \\
\end{bmatrix}
\begin{bmatrix}
5 & -2 & -1 \\
-2 & 1 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]

(b) Observe that

\[
\begin{bmatrix}
1 & 7 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]
That means that the linear map $A: \mathbb{R}^3 \to \mathbb{R}^3$ leaves the vector $v \in \mathbb{R}^3$ fixed which has with respect to the basis $B$ the coordinate vector
\[
c_B(v) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]
(Compare this with note at the bottom of page 68 in the lecture notes.) But this is precisely the first vector
\[
b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}
\]
of the basis $B$.
Indeed (just for illustration):
\[
\begin{pmatrix} -13 & 7 & -16 \\ -28 & 15 & -35 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}
\]
Thus for every $v \in U := \text{span}\{b_1\}$ there exists an $a \in \mathbb{R}$ such that $v = ab_1$ and it holds
\[
Av = A(ab_1) = aAb_1 = ab_1 = v.
\]
Since $U$ is by construction 1-dimensional it follows that $U$ is the desired subspace.
(c) Observe that
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}
\]
That means that the linear map $A: \mathbb{R}^3 \to \mathbb{R}^3$ maps the vector $u \in \mathbb{R}^3$ which has with respect to the basis $B$ the coordinate vector
\[
c_B(u) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
to the vector $2u$. But this is precisely the third vector
\[
b_3 = \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}
\]
of the basis $B$.
Indeed (just for illustration):
\[
\begin{pmatrix} -13 & 7 & -16 \\ -28 & 15 & -35 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 2 \end{pmatrix}
\]
Apparently this means that $W := \text{span}\{u\}$ is such that $A(W) = W$ and $W \neq U$.
(3) (a) We have
\[
I(1) = 2 \quad I(x) = 0
\]
\[
I(x^2) = \frac{2}{3} \quad I(x^3) = 0
\]
\[
I(x^4) = \frac{2}{5}
\]
and thus
\[ A := c_C^B(I) = \left( \begin{array}{cccc} 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 5 \end{array} \right) \]

(b) We have to solve the linear equation
\[ 2x_1 + \frac{2}{3}x_3 + \frac{2}{5}x_5 = 0 \]
which is equivalent with
\[ x_1 + \frac{1}{3}x_3 + \frac{1}{5}x_5 = 0. \]

Using the standard tools for solving this linear equation we obtain for example the following basis of the solution space
\[ K := \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \\ \end{pmatrix} \right\} \]
and \( \ker A = \text{span} K \).

(c) Observe that \( \ker I = \text{ker} A \). Thus
\[ L := \{ x, 3x^2 - 1, x^3, 5x^4 - 1 \} \]
is a possible basis of \( \ker I \).
In other words, every polynomial \( f \) of the form
\[ f(x) = 5a_4x^4 + a_3x^3 + 3a_2x^2 + a_1x - (a_4 + a_2) \]
with \( a_1, a_2, a_3, a_4 \in \mathbb{R} \) is in the kernel of \( I \) and vice versa every polynomial \( f \) in the kernel of \( I \) is of this form for some numbers \( a_1, a_2, a_3, a_4 \in \mathbb{R} \).

(4) (a) The transition matrix from the basis \( C \) to \( B \) is coordinate matrix of the identity map \( \text{id} : F^n \to F^n \) with respect to the bases \( B \) and \( C \), that is \( c_C^B(\text{id}) \). Denote the standard basis of \( F^n \) by \( E \). Then
\[ c_C^B(\text{id}) = c_C^B(\text{id} \circ \text{id}) = c_E^C(\text{id})c_E^B(\text{id}) = (c_E^C(\text{id}))^{-1}c_E^B(\text{id}) = T^{-1}S. \]
Thus the transition matrix from \( C \) to \( B \) is given by \( T^{-1}S \).

(b) The transition matrix \( S \) from \( B \) to the standard basis of \( \mathbb{R}^3 \) and the transition matrix \( T \) from \( C \) to the standard basis of \( \mathbb{R}^3 \) are the matrices
\[ S := \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]
We have to invert the matrix \( T \) (and we use the algorithm described on page 87 in the lecture notes):
\[
\begin{pmatrix}
2 & 0 & -1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & -3 & 1 & -2 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 \\
1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0
\end{pmatrix}
\]
Thus
\[ T^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}. \]

Then the transition matrix from basis \( C \) to the basis \( B \) is
\[ T^{-1}S = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ -1 & 0 & 1 \\ \frac{1}{3} & 1 & \frac{2}{3} \end{pmatrix}. \]

(c) The transition matrix \( S \) from \( B' \) to the standard basis of \( \mathbb{R}^3 \) and the transition matrix \( T \) from \( C' \) to the standard basis of \( \mathbb{R}^3 \) are the matrices
\[ S := \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 5 & 2 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 4 & 1 \end{pmatrix}. \]

We have to invert the matrix \( T \) (and we use the algorithm described on page 87 in the lecture notes):

\[
\begin{array}{ccc|ccc}
1 & -1 & 2 & 1 & 0 & 0 \\
1 & 2 & -1 & 0 & 1 & 0 \\
0 & 4 & 1 & 0 & 0 & 1 \\
\hline
1 & -1 & 2 & 1 & 0 & 0 \\
0 & 3 & -3 & -1 & 1 & 0 \\
0 & 4 & 1 & 0 & 0 & 1 \\
\hline
1 & -1 & 2 & 1 & 0 & 0 \\
0 & 1 & 4 & 1 & -1 & 1 \\
0 & 3 & -3 & -1 & 1 & 0 \\
\hline
1 & 0 & 6 & 2 & -1 & 1 \\
0 & 1 & 4 & 1 & -1 & 1 \\
0 & 0 & -15 & -4 & 4 & -3 \\
\hline
1 & 0 & 6 & 2 & -1 & 1 \\
0 & 1 & 4 & 1 & -1 & 1 \\
0 & 0 & -15 & -4 & 4 & -3 \\
\hline
1 & 0 & 0 & \frac{2}{5} & -\frac{4}{5} & \frac{1}{5} \\
0 & 1 & 0 & \frac{2}{15} & \frac{4}{15} & \frac{7}{5} \\
0 & 0 & 1 & \frac{2}{15} & \frac{4}{15} & \frac{7}{5} \\
\end{array}
\]

Thus
\[ T^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{15} & \frac{1}{15} & \frac{1}{5} \\ \frac{4}{15} & -\frac{4}{15} & \frac{1}{5} \end{pmatrix}. \]

Then the transition matrix from basis \( C' \) to the basis \( B' \) is
\[ T^{-1}S = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{15} & \frac{1}{15} & \frac{1}{5} \\ \frac{4}{15} & -\frac{4}{15} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{11}{15} & \frac{2}{5} \\ \frac{2}{15} & \frac{14}{15} & \frac{2}{5} \\ \frac{7}{15} & \frac{23}{15} & \frac{2}{5} \end{pmatrix}. \]

\((5^*)\) (a) There is actually not much to say here. A set of two vectors is linearly independent if neither of those vectors is a multiple of the other vectors. Now neither
\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{nor} \quad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
is a multiple of the other matrix and thus \{I, E\} is linearly independent. Similarly neither
\[ X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \] nor \[ Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
is a multiple of the other matrix and thus \{X, Y\} is linearly independent.

(b) If \( A \in U \) then there exists real numbers \( a_1, a_2 \) such that
\[ A = a_1 I + a_2 E. \]
Similarly if \( B \in U \) then there exists real numbers \( b_1, b_2 \) such that
\[ B = b_1 I + b_2 E. \]
Then – using the calculation rules of an \( \mathbb{R} \)-algebra – we have
\[
AB = (a_1 I + a_2 E)(b_1 I + b_2 E) \\
= a_1 I(b_1 I + b_2 E) + a_2 E(b_1 I + b_2 E) \\
= a_1 b_1 I + a_1 b_2 E + a_2 b_1 E + a_2 b_2 E^2 
\]
and since \( E^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \) we get finally
\[
= (a_1 b_1 - a_2 b_2)I + (a_1 b_2 + a_2 b_1)E \in U. 
\]
That is, \( U \) is closed under matrix multiplication.

**Note.** Using the fact that \( U \) is a linear subspace of the \( \mathbb{R} \)-algebra \( M_2(\mathbb{R}) \) and closed under the matrix multiplication and containing the identity matrix it follows that \( U \) is an \( \mathbb{R} \)-algebra.

(c) If \( A \in W \) then there exists real numbers \( a_1, a_2 \) such that
\[ A = a_1 X + a_2 Y. \]
Similarly if \( B \in W \) then there exists real numbers \( b_1, b_2 \) such that
\[ B = b_1 X + b_2 Y. \]
Then – using the calculation rules of an \( \mathbb{R} \)-algebra – we have
\[
AB = (a_1 X + a_2 Y)(b_1 X + b_2 Y) \\
= a_1 X(b_1 X + b_2 Y) + a_2 Y(b_1 X + b_2 Y) \\
= a_1 b_1 X^2 + a_1 b_2 XY + a_2 b_1 YX + a_2 b_2 Y^2 
\]
and since \( X^2 = X, Y^2 = Y \) and \( XY = YX = 0 \) we get finally
\[
= a_1 b_1 X + a_2 b_2 Y \in W. 
\]
That is, also \( W \) is closed under matrix multiplication. Furthermore we have that \( I = X + Y \) and thus \( I \in W \).

**Note.** Using the fact that \( W \) is a linear subspace of the \( \mathbb{R} \)-algebra \( M_2(\mathbb{R}) \) and closed under the matrix multiplication and containing the identity matrix it follows that \( W \) is an \( \mathbb{R} \)-algebra.

(d) Assume towards a contradiction that there exists a homomorphism \( \varphi: U \to W \) of \( \mathbb{R} \)-algebras. Since \( I \) is the identity of the matrix multiplication we have that necessarily \( \varphi(I) = I \). Furthermore there must exist real numbers \( a, b \) such that \( \varphi(E) = aX + bY \).
We want to calculate \( \varphi(E^2) \). On one hand we have
\[
\varphi(E^2) = \varphi(-I) = -\varphi(I) = -I = -X - Y
\]
and on the other hand we get
\[ \varphi(E^2) = \varphi(E)\varphi(E) = (aX + bY)(aX + bY) \]
\[ = a^2X + b^2Y. \]

But from this follows (since \( X \) and \( Y \) forms a basis of \( W \)) that \( a^2 = -1 \) and \( b^2 = -1 \) which is impossible! Therefore the assumption that there exists a homomorphism of \( \mathbb{R} \)-algebras from \( U \) to \( W \) is shown to be wrong. Therefore there cannot exists any such homomorphism of \( \mathbb{R} \)-algebras.

**Note.** In particular this means that \( U \) and \( W \) are not isomorphic as \( \mathbb{R} \)-algebras and this even though

(i) they are isomorphic as \( \mathbb{R} \)-vector spaces,

(ii) they are linear subspaces of the same \( \mathbb{R} \)-algebra \( M_2(\mathbb{R}) \) and

(iii) they use the same multiplication as in \( M_2(\mathbb{R}) \).

This shows that being isomorphic as algebras is a much stronger condition as being isomorphic as vector spaces. It also shows that there are \( F \)-algebras where there do not exists any homomorphism of algebras from one to the other algebra. In contrary there exists always the trivial map between any \( F \)-vector spaces. And between non-trivial \( F \)-vector spaces there exists always non-trivial linear maps (which follows from Proposition 3.5).