Department of Mathematics and Statistics  
Linear Algebra and Matrices  
Exercise 10  
19.3.2007

To be discussed on Friday, April 27th. Exercises with a star (*) give extra points.

(1) Proposition 3.59 on page 87 describes an algorithm how to check whether an \( n \times n \)-matrix \( A \) is invertible or not and how to determine at the same time its inverse \( A^{-1} \) in case it is invertible. See also the example on page 87 which illustrates the algorithm.\(^1\)

(a) Determine the inverse of the \( 2 \times 2 \)-matrix
\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\]

(b) Determine the inverse of the \( 3 \times 3 \)-matrix
\[
\begin{pmatrix}
-13 & 7 & -16 \\
-28 & 15 & -35 \\
0 & 0 & -2
\end{pmatrix}
\]

(c) Determine the inverse of the \( 3 \times 3 \)-matrix
\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 5 & 7 \\
0 & 0 & 1
\end{pmatrix}
\]

(2) Consider the endomorphism \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) given by
\[
A := 
\begin{pmatrix}
-13 & 7 & -16 \\
-28 & 15 & -35 \\
0 & 0 & -2
\end{pmatrix}
\]

(a) What is the coordinate matrix \( A' \) of \( A \) with respect to the basis
\[
B := \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}
\]

of \( \mathbb{R}^3 \)?

(b) Use the matrix \( A' \) to find a 1-dimensional subspace \( U \) of \( \mathbb{R}^3 \) such that \( Av = v \) for every vector \( v \in U \).\(^2\)

(c) There exists another 1-dimensional subspace \( W \neq U \) such that
\[\text{im}(A|_W) = W^3\]
Which one?

Remark. This exercise shows that the use of the basis transformations is to understand linear maps better.

(3) Consider the \( \mathbb{R} \)-vector space \( C^0([-1, 1]) \) of all continuous functions from the interval \([-1, 1]\) to \( \mathbb{R} \). Let \( U \) be the 5-dimensional subspace of \( C^0([-1, 1]) \) consisting of all polynomials of degree less or equal to 4. That is
\[
U := \{ a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 : a_i \in \mathbb{R} \}.
\]

\(^1\)You will need the a recent version of the lecture notes.

\(^2\)That is the restriction of \( A \) to \( U \) is the identity on \( U \), in symbols \( A|_U = \text{id}_U \).

\(^3\)Note that \( \text{im}(A|_W) = \{ Av : v \in W \} \), that is \( \text{im}(A|_W) \) is the image of the restriction of the linear map \( A \) to the subspace \( W \).
Consider the linear map
\[ I: U \to \mathbb{R}, f \mapsto I(f) := \int_{-1}^{1} f(x)dx. \]

(a) Determine the coordinate matrix \( A \) of \( I \) with respect to the basis
\[ B = (1, x, x^2, x^3, x^4) \]
and the standard basis \( C = (1) \) of \( \mathbb{R} \).

(b) Determine \( \ker A \) of the linear map \( A: \mathbb{R}^5 \to \mathbb{R} \).

(c) Use this to determine \( \ker I \), that is the set of all polynomials \( f \) of degree less or equal to 4 such that \( \int_{-1}^{1} f(x)dx = 0 \).

Remark. Of course there exists much simpler approaches to solve this “artificial” problem without the use of Linear Algebra...

(4) (a) Let \( F \) be a field and consider the \( F \)-vector space \( F^n \). Let \( B \) and \( C \) be arbitrary bases of \( F^n \). Let \( S \) be the transition matrix from canonical standard basis of \( F^n \) to \( B \) and let \( T \) be the transition matrix from the canonical standard basis of \( F^n \) to \( C \).

Derive a formula for the transition matrix from the basis \( C \) to the basis \( B \) using the matrices \( S \) and \( T \).

(b) Consider the bases
\[ B := \left( \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \\ 3 \\ 4 \\ 5 \\ 1 \\ 1 \end{pmatrix} \right) \]

and
\[ C := \left( \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \]

of the vector space \( \mathbb{R}^3 \).

Compute the transition matrix from the basis \( C \) to the basis \( B \).

(c) Consider the bases
\[ B' := \left( \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 5 \\ 4 \\ 2 \\ 1 \\ -1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \]

and
\[ C' := \left( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right) \]

of the vector space \( \mathbb{R}^3 \).

Compute the transition matrix from the basis \( C' \) to the basis \( B' \).

(5*) This exercise gives a concrete example of two \( \mathbb{R} \)-algebras which are isomorphic as vector spaces but not isomorphic as \( \mathbb{R} \)-algebras.

Consider the vector space of all real \( 2 \times 2 \)-matrices \( M_2(\mathbb{R}) = \mathbb{R}^{2 \times 2} \). We know that this vector space is in a natural way an \( \mathbb{R} \)-algebra. Define the following elements in \( M_2(\mathbb{R}) \):
\[ I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
(a) Show that the sets \( \{I, E\} \) and \( \{X, Y\} \) are linear independent subsets of \( M_2(\mathbb{R}) \).

Thus \( U := \text{span}\{I, E\} \) and \( W := \text{span}\{X, Y\} \) are both 2-dimensional subspaces of \( M_2(\mathbb{R}) \).

(b) Show that \( U \) is closed under matrix multiplication, that is if \( A, B \in U \) then also their product \( AB \in U \).

(c) Show that also \( W \) is closed under the matrix multiplication and that furthermore \( I \in W \).

Now since \( M_2(\mathbb{R}) \) is an \( \mathbb{R} \)-algebra it is not difficult to see that from (b) and (c) follows that both \( U \) and \( W \) are \( \mathbb{R} \)-algebras, too.

(d) Show that there cannot exists any homomorphism \( \varphi: U \to W \) of \( \mathbb{R} \)-algebras.

In particular the result of (d) means that \( U \) and \( W \) are not isomorphic \( \mathbb{R} \)-algebras. And this even though they are isomorphic as \( \mathbb{R} \)-vector spaces.

Hints: In (b) write \( A = a_1I + a_2E \) and \( B = b_1I + b_2E \) for some \( a_1, a_2, b_1, b_2 \in \mathbb{R} \). Conclude then that the product \( AB \) is a linear combination of \( I \) and \( E \). Solve (c) with a similar approach. In (d) assume towards a contradiction that there exists a homomorphism \( \varphi: U \to W \) of \( \mathbb{R} \)-algebras. Then we know that \( \varphi(I) = I \) and \( \varphi(E) = aX + bY \) for some \( a, b \in \mathbb{R} \). There are two possible ways to calculate \( \varphi(EE) \) and this will lead to a contradiction.

Remark. \( U \cong \mathbb{C} \) as \( \mathbb{R} \)-algebras.