(1) The extended coefficient matrix for the given system of linear equations is:

\[
\begin{pmatrix}
13 & 13 & 5 & 9 & b_1 \\
1 & 0 & -1 & 3 & b_2 \\
5 & 3 & 0 & 2 & b_3 \\
0 & 7 & 8 & -4 & b_4
\end{pmatrix}
\]

Using elementary column transformations we can transform this matrix into:

\[
\begin{pmatrix}
11 & 0 & 0 & -46 & -24b_2 + 7b_3 - 3b_4 \\
0 & 11 & 0 & 84 & 40b_2 - 8b_3 + 5b_4 \\
0 & 0 & 11 & -79 & -35b_2 + 7b_3 - 3b_4 \\
0 & 0 & 0 & 0 & b_1 - 3b_2 - 2b_3 - b_4
\end{pmatrix}
\]

\[(*)\]

(a) In the case that \( b_1 = b_2 = b_3 = b_4 = 0 \) the matrix \((*)\) becomes

\[
\begin{pmatrix}
11 & 0 & 0 & -46 & 0 \\
0 & 11 & 0 & 84 & 0 \\
0 & 0 & 11 & -79 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Thus all the solutions of the system of linear equations is given by

\[x := t \cdot \begin{pmatrix} 46 \\ -84 \\ 79 \\ 11 \end{pmatrix} \quad (t \in \mathbb{R}).\]

(b) In the case \( b_1 = 0, b_2 = 1, b_3 = 2 \) and \( b_4 = -7 \) the matrix \((*)\) becomes

\[
\begin{pmatrix}
11 & 0 & 0 & -46 & 11 \\
0 & 11 & 0 & 84 & -11 \\
0 & 0 & 11 & -79 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

We see that the nonhomogeneous system of linear equations is solvable and that a special solution is given by

\[x_0 := \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\]

Thus all the solutions of the system of linear equations is given by

\[x := t \cdot \begin{pmatrix} 46 \\ -84 \\ 79 \\ 11 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad (t \in \mathbb{R}).\]
(c) In the case that $b_1 = 1$, $b_2 = b_3 = b_4 = 0$ the matrix (*) becomes

$$
\begin{pmatrix}
11 & 0 & 0 & -46 & 0 \\
0 & 11 & 0 & 84 & 0 \\
0 & 0 & 11 & -79 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

One can see that the given nonhomogeneous system of linear equations is not solvable in this case because the last line represents the nonsolvable equation $0 = 1$.

(2) (a) We have:

$$0x = 0x \\
\Rightarrow 0x = (0 + 0)x \quad \text{(A3)} \\
\Rightarrow 0x = 0x + 0x \quad \text{(D)} \\
\Rightarrow 0x - 0x = 0x + 0x - 0x \\
\Rightarrow 0 = 0x \quad \text{(A4)}$$

(b) We begin with the equality $0 = 0y$ which we know is true from the above calculation.

$$0 = 0y \\
\Rightarrow 0 = (x - x)y \quad \text{(A4)} \\
\Rightarrow 0 = xy + (-x)y \quad \text{(D)} \\
\Rightarrow -(xy) + 0 = -(xy) + xy + (-x)y \\
\Rightarrow -(xy) + 0 = 0 + (-x)y \quad \text{(A4)} \\
\Rightarrow -(xy) = -x(y) \quad \text{(A3)}$$

(c) Assume that $xy = 0$. Assume towards a contradiction that neither $x = 0$ nor $y = 0$.

$$xy = 0 \\
\Rightarrow (1/x)xy = (1/x)0 \quad \text{(M4) + (2a)} \\
\Rightarrow 1y = 0 \quad \text{(M3)} \\
\Rightarrow y = 0 \\
\Rightarrow (1/y)y = (1/y)0 \quad \text{(M4) + (2a)} \\
\Rightarrow 1 = 0$$

But now the last row contradicts to the requirement $0 \neq 1$ of the axiom (M3). Thus our assumption that $x \neq 0$ and $y \neq 0$ is wrong. This means that $x = 0$ or $y = 0$ is true.

(d) $(xy)(1/x)(1/y) = x(1/x)y(1/y) = 1 \Rightarrow (1/x)(1/y) = 1/(xy)$.

(e) $(xv + yu)/(yu) = (xv/yv) + (yu/yv) = xy(1/y)(1/v) + yu(1/y)(1/v) = x/v + u/v$.

(3) We have:

$$x = x \cdot 1 \quad \text{(property of 1)} \\
= x \cdot (1 + 0) \quad \text{(property of 0)} \\
= x \cdot 1 + x \cdot 0 \quad \text{(distributive law)} \\
= x \cdot 1 + x \cdot 1 \quad \text{(assumption 0 = 1)} \\
= x + x \quad \text{(property of 1)}$$

Thus $x = x + x$ and adding $-x$ to both sides of this equation yields $0 = x$. 

(4) Recall the special magic squares introduced in exercise (4*) of the previous exercise sheet:

\[ e_1 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}. \]

Looking at the special constraints for the magic squares \( x, y \) and \( z \) in the given equations and using the form of the magic squares \( e_1, e_2 \) and \( e_3 \) we conclude the following:

- Since \( x_5 = 0 \) it follows that
  \[ x = r_1 e_1 + r_2 e_2 + r_3 e_3 \]
  if and only if \( r_1 = 0 \). That is we have that
  \[ x = r_2 e_2 + r_3 e_3 \]
  for some numbers \( r_2, r_3 \in \mathbb{R} \).

- Since \( y_3 = y_7 = 4 \) it follows that
  \[ y = s_1 e_1 + s_2 e_2 + s_3 e_3 \]
  if and only if \( s_1 = 4 \) and \( s_3 = 0 \). That is, we have that
  \[ y = 4 e_1 + s_2 e_2 \]
  for some number \( s_2 \in \mathbb{R} \).

- Since \( z_4 = 1 \) and \( z_8 = 3 \) it follows that
  \[ z = t_1 e_1 + t_2 e_2 + t_3 e_3 \]
  if and only if \( t_1 - t_3 = 2 \) and \( t_2 = 1 \). That is, we have that
  \[ z = t_1 e_1 + e_2 + (t_1 - 2) e_3 \]
  for some number \( t_1 \in \mathbb{R} \).

Then

\[ x + y = r_2 e_2 + r_3 e_3 + 4 e_1 + s_2 e_2 \]
\[ = 4 e_1 + (r_2 + s_2) e_2 + r_3 e_3 \]
\[ = z \]

if and only if

\[ t_1 = 4 \]
\[ 1 = r_2 + s_2 \]
\[ t_1 - 2 = r_3 \]

From \( t_1 = 4 \) follows that \( r_3 = 2 \). And if we set \( r_2 := r \), then the above equalities are satisfied if and only if \( s_2 = 1 - r \). Thus we have that all possible solution to the equation \( x + y = z \) of magic squares with the given
constraints are

\[
x = re_2 + 2e_3 = \\
= \begin{pmatrix}
  r & -r + 2 & -2 \\
  -r - 2 & 0 & r + 2 \\
  2 & r - 2 & -r
\end{pmatrix}
\]

\[
y = 4e_1 + (1 - r)e_2 = \\
= \begin{pmatrix}
  5 - r & 3 + r & 4 \\
  3 + r & 4 & 5 - r \\
  4 & 5 - r & 3 + r
\end{pmatrix}
\]

\[
z = 4e_1 + e_2 + 2e_3 = \\
= \begin{pmatrix}
  5 & 5 & 2 \\
  1 & 4 & 7 \\
  6 & 3 & 3
\end{pmatrix}
\]