In order to show the equality "\( A \cap (B+C) = (A \cap B) + C \)" we have to show the two inclusions "\( A \cap (B+C) \subset (A \cap B) + C \)" and "\( A \cap (B+C) \supset (A \cap B) + C \)".

"\( \subset \)" Let \( x \in A \cap (B+C) = (A \cap B) + C \). Then \( x \in B + C \) and thus there exists \( y \in B \) and \( z \in C \) such that \( x = y + z \). On the other hand we have that also \( x \in A \) and therefore \( x = y + z \in A \), too. Since by assumption \( C \subset A \) we have that \( z \in A \). Therefore \( y = z - x \in A \) since \( A \) is a vector space. Then \( y \in A \cap B \) and \( x = y + z \in (A \cap B) + C \). This shows the inclusion \( A \cap (B+C) \subset (A \cap B) + C \).

"\( \supset \)" Let \( x \in (A \cap B) + C \). Therefore there exist \( y \in A \cap B \) and \( z \in C \) such that \( x = y + z \). Clearly \( x \in B + C \). Since \( C \subset A \) we have that \( z \in A \), too. Since \( A \) is a vector space we get that then also \( x = y + z \in A \). Thus \( x \in A \cap (B + C) \) and this concludes the proof of the inclusion \( (A \cap B) + C \subset A \cap (B+C) \).

(2) (a) Let \( v \in M \). Then \( v = 1 \cdot v \) is a linear combination of elements in \( M \). Therefore \( v \in \text{span} \ M \). That is \( M \subset \text{span} \ M \).

(b) Let \( v \in \text{span} \ M \). Then \( x \) is a linear combination of vectors of \( M \). But since \( M \subset M' \) this means that \( v \) is also a linear combination of vectors \( M' \) (the same linear combination will do!). Therefore \( v \in \text{span} \ M' \).

This shows \( \text{span} \ M \subset \text{span} \ M' \).

(c) "\( \Rightarrow \)" If \( M = \text{span} \ M \) then \( M \) is a linear subspace since we know that \( \text{span} \ M \) is a linear subspace (Proposition 2.12).

"\( \Leftarrow \)" Now assume that \( M \) is a linear subspace. We know already from the first part that in \( M \subset \text{span} \ M \). Therefore we need only to show the inclusion in the opposite direction. Thus let \( v \in \text{span} \ M \), that is we can write \( v \) as a linear combination

\[
v = \sum_{u \in M} a_u u
\]

of vectors of \( M \). But since \( M \) is a linear subspace of \( V \) we have that \( v \in M \). Therefore \( \text{span} \ M \subset M \) and altogether equality holds.

(d) Since we know that \( \text{span} \ M \) is a linear subspace of \( V \) we know by the previous part that \( \text{span}(\text{span}(M)) = \text{span} \ M \).

(e) We need to show the inclusion in both directions:

"\( \subset \)" Let \( v \in \text{span}(M \cup M') \). Then there exists a \( x \in F(M \cup M') \) such that

\[
v = \sum_{u \in M \cup M'} x(u) u
\]

and thus

\[
v = \sum_{u \in M} x(u) u + \sum_{u \in M' \setminus M} x(u) u
\]

\( \in \text{span} \ M + \text{span} \ M' \)

since the restriction of \( x \) to \( M \) is a vector of \( F(M) \) and the restriction of \( x \) to \( M' \setminus M \) is a vector for \( F(M') \). Therefore \( \text{span}(M \cup M') \subset \text{span} \ M + \text{span} \ M' \).
Let \( v \in \text{span } \mathcal{M} + \text{span } \mathcal{M}' \). Then there exists a \( x \in F(M) \) and \( x' \in F(M') \) such that

\[
v = \sum_{u \in \mathcal{M}}' x(u)u + \sum_{u \in \mathcal{M}'}' x'(u)u
\]

and therefore by linearity

\[
v = \sum_{u \in \mathcal{M} \cup \mathcal{M}'} (x + x')(u)u
\]

\( \in \text{span}(\mathcal{M} \cup \mathcal{M}') \)

since \( x + x' \in F(M \cup M') \). Thus \( \text{span } \mathcal{M} + \text{span } \mathcal{M}' \subset \text{span}(\mathcal{M} \cup \mathcal{M}') \).

(3) • Let us first determine \( U \cap W \). If \( x \in U \cap W \) then there exists \( t_1, t_2, t_3, t_4 \in \mathbb{R} \) such that \( x = t_1v_1 + t_2v_2 = t_3v_3 + t_4v_4 \). We are interested in all possible values for \( t_1, t_2, t_3 \) and \( t_4 \) which satisfy this equation.

That is, we have to solve the homogeneous system of linear equations

\[
t_1v_1 + t_2v_2 - t_3v_3 - t_4v_3 = 0.
\]

Its simple coefficient matrix is:

\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
1 & -1 & 1 & -1
\end{pmatrix}
\]

Using elementary row transformations we can transform this matrix into the matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

We get that all the solutions of the equation are given by \( t_1 = t_2 = t_3 = t_4 = t \in \mathbb{R} \).

Therefore \( U \cap W \) is equal to

\[
U \cap W = \left\{ t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}.
\]

• We claim that \( U + W = \mathbb{R}^3 \). Therefore we need to show that for every \( x \in \mathbb{R}^3 \) there exists numbers \( t_1, t_2, t_3, t_4 \in \mathbb{R} \) such that

\[
x = t_1v_1 + t_2v_2 + t_3v_3 + t_4v_4.
\]

This is a nonhomogeneous system of 3 linear equations in the 4 unknown variables \( t_1, t_2, t_3 \) and \( t_4 \). Its extended coefficient matrix is:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & x_1 \\
0 & 1 & 0 & 1 & x_2 \\
1 & -1 & -1 & 1 & x_3
\end{pmatrix}
\]

Using elementary row transformations we can transform it into

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & \frac{1}{2}(x_1 + x_2 + x_3) \\
0 & 1 & 0 & 1 & x_2 \\
0 & 0 & 1 & -1 & \frac{1}{2}(x_1 - x_2 - x_3)
\end{pmatrix}
\]

Therefore we can see that we can always find \( t_1, t_2, t_3, t_4 \in \mathbb{R} \) such that

\[
x = t_1v_1 + t_2v_2 + t_3v_3 + t_4v_4.
\]
Consider the case
Let (a)
Assume that there exists a linear combination of the zero vector by elements
\[ 0 = a_0 x^0 + \ldots + a_n x^n. \]
Since the zero vector of \( C \) of \( v \) in \( \mathbb{R}^3 \), we have shown that \( \mathbb{R}^3 \subset U + W \) and equality holds since \( U + W \subset \mathbb{R}^3 \).

- \( U \) and \( W \) are two planes passing through the origin which intersect in a line \( U \cap W \) (passing through the origin, too) and which “span” the whole \( \mathbb{R}^3 \).

(4) Assume that there exists a linear combination of the zero vector by elements of \( M \). Then there exists a minimal \( n \in \mathbb{N} \) such that we can write the 0 vector in \( C^0(\mathbb{R}) \) as
\[ a_0 p_0 + \ldots + a_0 p_0 = 0. \]
Since the zero vector of \( C^0(\mathbb{R}) \) is the constant zero-function we have from the above equality that
\[ a_0 x^0 + \ldots + a_n x^n = 0 \]
for every \( x \in \mathbb{R} \).
We have to distinguish two cases \( n = 0 \) and \( n > 0 \).

(a) Consider first the case \( n = 0 \). Then \( a_0 = 0 \) and (a) reduces to the trivial linear combination.

(b) Thus we may assume that \( n > 0 \). Then due to the minimality of \( n \) we have that \( a_n \neq 0 \). Thus (**) says that \( (*) \) is a polynomial of degree \( n \) with infinite many roots. But this contradicts the Fundamental Theorem of the Algebra which states that \( (*) \) can have at most \( n \) roots.
Therefore this case is not possible.
Thus only the first case is possible and in this case (*) it follows that the zero vector of \( C^0(\mathbb{R}) \) can only be the trivial linear combination of vectors in \( M \).

(5*) Let \( f, g \in U \) and \( a \in F \). Then \( (f + g)(x_0) = f(x_0) + g(x_0) = 0 + 0 = 0 \) and 
\[ (a f)(x_0) = a f(x_0) = a 0 = 0. \]
Thus \( f + g \in U \) and \( a f \in U \). Clearly \( U \) is not empty since \( 0 \in U \). Thus it follows by the subspace criterion that \( U \) is a subspace of \( F^M \).
Let \( f, g \in W \) and \( a \in F \). Then for every \( x, y \in M \) we have
\[ (f + g)(x) = f(x) + g(x) \]
\[ = f(y) + g(y) \]
\[ = (f + g)(y) \]
and
\[ (a f)(x) = a f(x) \]
\[ = a f(y) \]
\[ = (a f)(y). \]
Therefore \( f + g \in W \) and \( a f \in W \). Clearly \( W \) is not empty either since \( 0 \in W \). Thus we can apply the subspace criterion and we get that \( W \) is a subspace of \( F^M \).

(b) Let \( f \in U \cap W \). Since \( f \in U \) we have that \( f(x_0) = 0 \) and since \( f \in W \) we have that \( f(x) = f(y) \) for every \( x, y \in M \). In particular \( f(x) = f(x_0) = 0 \) for every \( x \in M \) and thus \( f = 0 \). Therefore \( U \cap W = \{0\} \).

(c) Let \( h \in F^M \) be an arbitrary map. Define maps \( f, g \in F^M \) by \( f(x) := h(x) - h(x_0) \) and \( g(x) := h(x_0) \) for every \( x \in M \). Then \( f \in U \) and \( g \in W \) and by construction we have \( h = f + g \). Thus \( h \in U + W \) and this shows that \( F^M \subset U + W \). Equality holds since \( U + W \subset F^M \).