(1) (a) The set $M_1$ is not empty since apparently $0 \in M_1$. If $x, y \in M_1$, then $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$. But then also $x_1 + y_1 = x_2 + y_2 = x_3 + y_3$ and therefore $x + y \in M_1$. Similar, if $x \in M_1$ and $a \in F$, then from $x_1 = x_2 = x_3$ follows that $ax_1 = ax_2 = ax_3$ and therefore $ax \in M_1$. Thus $M_1$ is a subspace by the subspace criterion.

(b) The set $M_2$ is not empty since apparently $0 \in M_2$. If $x, y \in M_2$, then $x_3 = 0$ and $y_3 = 0$. But then also $x_3 + y_3 = 0$ and therefore $x + y \in M_2$. Similar, if $x \in M_2$ and $a \in F$, then from $x_3 = 0$ follows that $ax_3 = 0$ and herefore $ax \in M_2$. Thus $M_2$ is a subspace by the subspe criterion.

(c) The set $M_3$ is not empty since apparently $0 \in M_3$. If $x, y \in M_3$, then $x_1 = x_2 - x_3$ and $y_1 = y_2 - y_3$. But then also $x_1 + y_1 = x_2 + y_2 - (x_3 + y_3)$ and therefore $x + y \in M_3$. Similar, if $x \in M_3$ and $a \in F$, then from $x_1 = x_2 - x_3$ follows that $ax_1 = ax_2 - ax_3$ and herefore $ax \in M_3$. Thus $M_3$ is a subspace by the subspace criterion.

(d) The set $M_4$ is not a subspace in of $V$. For example $0 \notin M_4$.

(2) (a) Set $U_1 := \text{span} M_1$. Since $M_1$ is a subspace of $V$ it follows that $U_1 = M_1$. Now $x \in M_1$ if and only if

\[
\begin{align*}
x_1 - x_2 &= 0 \\
x_2 - x_3 &= 0
\end{align*}
\]

and this system of linear equation is equivalent with

\[
\begin{align*}
x_1 - x_3 &= 0 \\
x_2 - x_3 &= 0
\end{align*}
\]

We may chose one variable freely in $F$ – say $x_3 := t$ – and then it follows that the system of linear equation is satisfied if and only if $x_1 = t$ and $x_2 = t$. Thus

\[
U_1 = \{ \left( \begin{array}{c} 1 \\ 1 \\ t \end{array} \right) : t \in F \}
\]

and a basis of $U_1$ is apparently given by – for example –

\[
\{ \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right) \}
\]

(b) Set $U_2 := \text{span} M_2$. Since $M_2$ is a subspace of $V$ it follows that $U_2 = M_2$. Now $x \in M_2$ if and only if

\[x_3 = 0.\]

Thus we may chose $x_1$ and $x_2$ freely in $F$ – say $x_1 := s$ and $x_2 = t$. Then apparently

\[
U_2 = \{ \left( \begin{array}{c} s \\ t \\ 0 \end{array} \right) : s, t \in F \}
\]

and a basis of $U_2$ is apparently given by – for example –

\[
\{ \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \}
Set $U_3 := \text{span} M_3$. Since $M_3$ is a subspace of $V$ it follows that $U_3 = M_3$. Now $x \in M_3$ if and only if
\[ x_1 - x_2 + x_3 = 0. \]
Thus we may chose $x_2$ and $x_3$ freely in $F$—say $x_2 := s$ and $x_3 = t$—and then it follows that the linear equation above is satisfied if and only if $x_1 = s - t$. Then apparently
\[ U_3 = \{ (s-t) : s, t \in F \} \]
and a basis of $U_3$ is apparently given by—for example—
\[ \left\{ \left( \frac{1}{0} \right), \left( \frac{-1}{t} \right) \right\} \]
Set $U_4 := \text{span} M_4$. We have that the following vectors are elements in $M_4$:
\[ \left( \frac{0}{0} \right), \left( \frac{1}{1} \right), \left( \frac{0}{1} \right) \]
We claim first that this system of vectors is linear independent. Assume that $a_1, a_2, a_3 \in F$ such that
\[ a_1 \left( \frac{1}{0} \right) + a_2 \left( \frac{1}{1} \right) + a_3 \left( \frac{1}{0} \right) = \left( \frac{a_1+a_2+a_3}{a_2} \right) = \left( \frac{0}{0} \right) \]
Then apparently $a_2 = a_3 = 0$ and thus necessarily also $a_1 = 0$. Thus by Proposition 2.20 it follows that the those elements are linear independent.
Next we claim that the system of vectors generates $V$. If $x \in V$ is an arbitrary vector and set $a_1 := x_1 - x_2 - x_3$, $a_2 := x_2$ and $a_3 := x_3$. Then apparently
\[ a_1 \left( \frac{1}{0} \right) + a_2 \left( \frac{1}{1} \right) + a_3 \left( \frac{1}{0} \right) = \left( \frac{a_1+a_2+a_3}{a_2} \right) = \left( \frac{x_2}{x_3} \right) \]
and this proves the second claim.
Thus
\[ \left\{ \left( \frac{1}{0} \right), \left( \frac{1}{1} \right), \left( \frac{1}{0} \right) \right\} \]
is a linear independent generating system of $V$ and it follows by Proposition 2.23 that it is a basis of $V$. Since this basis is a subset of $M_4$ it follows that $M_4$ generates the whole vector space $V$.

(3) \(\Rightarrow\): We assume that $U \cup W$ is a subspace of $V$ and want to show that either $U \subset W$ or $W \subset U$. We do this by assuming towards a contradiction that neither $U \subset W$ or $W \subset U$. Thus there exists a $x \in U$ such that $x \notin W$ and $y \in W$ such that $y \notin U$. Since $U \cup W$ is by assumption a subspace it follows that $z := x + y \in U \cup W$. Thus $z \in U$ or $z \in W$. If $z \in U$, then $y = z - x \in U$ since $U$ is a subspace of $V$, but this contradicts to the assumption $y \notin U$. Thus the remaining case $z \in W$ must be true. But likewise we get a contradiction since then $x = z - y \in W$ since $W$ is a subspace. Therefore the assumption that neither $U \subset W$ or $W \subset U$ is proven to be wrong and it must hold that either $U \subset W$ or $W \subset U$.

\(\Leftarrow\): If $U \subset W$ or $W \subset U$, then $U \cup W = W$ or $U \cup W = U$. Thus in both cases $U \cup W$ must be a subspace since $U$ and $W$ are assumed to be subspaces of $V$. 
(4) (a) First observe that \( f(0) = f(0 \cdot 0) = 0f(0) = 0 = -0 \) and thus both \( U \) and \( W \) contain the zero vector 0. In particular \( U \) and \( W \) are not empty.

If \( x, y \in U \) and \( a \in F \), then \( f(x + y) = f(x) + f(y) = x + y \) and \( f(ax) = af(x) = ax \) and thus \( x + y \in U \) and \( ax \in U \). By the subspace criterion it follows that \( U \) is a subspace of \( V \).

If \( x, y \in W \) and \( a \in F \), then \( f(x + y) = f(x) + f(y) = -x - y = -(x + y) \) and \( f(ax) = af(x) = a(-x) = -ax \) and thus \( x + y \in W \) and \( ax \in W \). By the subspace criterion it follows that \( W \) is a subspace of \( V \), too.

(b) Let \( x \in U \cap W \). Then \( x = f(x) = -x \). Thus \( x + x = (1 + 1)x = 0 \).

But since \( \text{char } F \neq 2 \) it follows that \( 1 + 1 \neq 0 \) and therefore necessarily \( x = 0 \) (see the calculation rule (45) on page 22 of the lecture notes).

Thus \( U \cap W = 0 \).

(c) Let \( x \in V \) be an arbitrary vector. Since \( 1 + 1 \neq 0 \) we can set \( c := 1/(1+1) \in F \). Define further

\[ y := cx + cf(x) \quad \text{and} \quad z := cx - cf(x). \]

Then

\begin{align*}
y + z &= cx + cf(x) + cx - cf(x) \\
&= cx + cx \\
&= c(x + x) \\
&= c(1 + 1)x \\
&= x.
\end{align*}

Here the last equality is true because \( c(1 + 1) = 1 \) in \( F \).

We claim that \( y \in U \) and \( z \in W \) (which then implies that \( x \in U + W \)). We verify these claims:

\begin{align*}
f(y) &= f(cx + cf(x)) \\
&= cf(x) + cf(f(x)) \\
&= cf(x) + cx \\
&= y
\end{align*}

This implies \( y \in U \).

\begin{align*}
f(z) &= f(cx - cf(x)) \\
&= cf(x) - cf(f(x)) \\
&= cf(x) - cx \\
&= -(cx - cf(x)) \\
&= -z
\end{align*}

This implies \( z \in W \).
We need to verify all the field axioms (see page 19f. of the lecture notes).

The field axioms (A1) to (A4) are apparently satisfied by the addition of vectors.

If \( x, y, z \in V \), then
\[
(x \cdot y) \cdot z = \left( \frac{x_1 y_1 - x_2 y_2}{x_1 y_2 + x_2 y_1} \right) \cdot \left( \frac{z_1}{z_2} \right) = \left( \frac{x_1 y_1 z_1 - x_2 y_2 z_2}{x_1 y_1 z_2 - x_2 y_2 z_2} \right) \cdot \left( \frac{z_1}{z_2} \right) = \left( \frac{x_1 (y_1 z_1 - y_2 z_2) - x_2 (y_2 z_1 + y_1 z_2)}{x_1 (y_1 z_2 + y_2 z_1) + x_2 (y_1 z_1 - y_2 z_2)} \right) = x \cdot (y \cdot z).
\]

That is the multiplication on \( V \) is associative and this verifies the field axiom (M1).

If \( x, y \in V \), then
\[
x \cdot y = \left( \frac{x_1 y_1 - x_2 y_2}{x_1 y_2 + x_2 y_1} \right) = \left( \frac{y_1 x_1 - y_2 x_2}{y_1 x_2 + y_2 x_1} \right) = y \cdot x
\]

That is the multiplication on \( V \) is commutative and this verifies the field axiom (M2).

Denote by "1" the element \( \left( \frac{1}{1} \right) \in V \). Then for any \( x \in V \) we have
\[
x \cdot 1 = \left( \frac{x_1 \cdot 1 - x_2 \cdot 0}{x_1 \cdot 0 + x_2 \cdot 1} \right) = x
\]
and due to the commutativity of the multiplication also \( 1 \cdot x = x \). Therefore 1 is indeed the identity element of the multiplication and field axiom (M3) is satisfied by the multiplication on \( V \).

Assume that \( x, y \in V \) such that \( x \cdot y = 1 \). This happens if and only if
\[
x_1 y_1 = x_2 y_2 = 1
\]
\[
x_2 y_1 + x_1 y_2 = 0
\]
We shall assume that \( 0 \neq x \in V \) is given and we want to know for what values of \( y \) this is satisfied. It is a nonhomogeneous system of 2 linear equations in the two unknown variables \( y_1 \) and \( y_2 \). Then by Proposition 1.8 we know that this system has a unique solution if the associated homogeneous system has only the trivial solution. But this is the case by Proposition 1.1 since \( \delta(x_1, -x_2, x_2, x_1) = x_1^2 + x_2^2 \neq 0 \) by assumption. Thus there exists a unique solution \( y \in V \) to the above system of linear equations. Due to commutativity we have that also \( y \cdot x = 1 \). Therefore every \( x \) has a (unique) multiplicative inverse and this verifies the field axiom (M4).
If $x, y, z \in V$, then
\[
(x + y) \cdot z = \left( \frac{(x_1 + y_1)z_1 - (x_2 + y_2)z_2}{(x_1 + y_1)z_2 + (x_2 + y_2)z_1} \right)
\]
\[
= \left( \frac{x_1z_1 - x_2z_2 + y_1z_1 - y_2z_2}{x_1z_2 + x_2z_1 + y_1z_2 + y_2z_1} \right)
\]
\[
= \left( \frac{x_1z_1 - x_2z_2}{x_1z_2 + x_2z_1} \right) + \left( \frac{y_1z_1 - y_2z_2}{y_1z_2 + y_2z_1} \right)
\]
\[
= x \cdot z + y \cdot z.
\]
and due to the commutativity of the multiplication we have
\[
x \cdot (y + z) = (y + z) \cdot x = y \cdot x + z \cdot x = x \cdot y + x \cdot z.
\]
Thus the distributive law holds and this verifies the last remaining field axiom (D).