(1) (a) In order to determine the dimension of $U$ and $W$ we need – for example – to find a basis of those vector spaces. We know that $\{v_1, v_2, v_3\}$ contains a subset which is a basis of $U$ since $\{v_1, v_2, v_3\}$ is a generating system. Since $v_1 + v_2 = v_3$ we have that $v_3 \in \text{span}\{v_1, v_2\}$. Therefore $\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3\} = U$ and thus also $\{v_1, v_2\}$ is a generating system of $U$. Apparently $\{v_1, v_2\}$ is also linear independent and thus $\{v_1, v_2\}$ is a linear independent generating system of $U$ and by Proposition 2.23 a basis of $U$. Since this basis contains precisely 2 vectors it follows that $\dim U = 2$.

Similarly we deduce the dimension of the linear subspace $W$. We know that the set $\{w_1, w_2, w_3\}$ contains a basis since it is a generating system of $W$. Since $w_1 + w_2 = w_3$ it follows that $w_3 \in \text{span}\{w_1, w_2\}$ and thus $\text{span}\{w_1, w_2\} = \text{span}\{w_1, w_2, w_3\} = W$. Therefore also $\{w_1, w_2\}$ is a generating system of $W$. Apparently this set is also linear independent and thus a basis by Proposition 2.23. It follows that $\dim W = 2$, too.

(b) An other way to determine the dimension of a subspace is to use Proposition 2.39. From the previous part we know that $\{v_1, v_2, w_1, w_2\}$ is a generating system of $U + W$. We have to determine the rank of the system of vectors $v_1, v_2, w_1, w_2$. We do this by using the result of Theorem 2.32. The coordinate matrix with respect to the standard basis of these vectors is

$$
\begin{pmatrix}
1 & 1 & 2 & 1 \\
0 & -2 & 1 & 0 \\
3 & 1 & 7 & 3 \\
-3 & 2 & -5 & -1
\end{pmatrix}
$$

Using elementary row transformations we can transform this matrix into the matrix

$$
\begin{pmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & 3 & 2 \\
0 & 0 & 7 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and apparently this means that the system $v_1, v_2, w_1, w_3$ has rank 3. Thus $\dim(U + W) = 3$.

(c) Now using the above results we can use the dimension formula of Theorem 2.43 to deduce the dimension of $U \cap W$. We have

$$
\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 2 + 2 - 3 = 1.
$$

(2) (a) Apparently $0 \in U$ and thus $U$ is not empty. Let $x, y \in U$ and $a \in \mathbb{R}$. Then

$$
(x_1 + y_1) + 2(x_2 + y_2) = (x_1 + 2x_2) + (y_1 + 2y_2)
= (x_3 + 2x_4) + (y_3 + 2y_4)
= (x_3 + y_3) + 2(x_4 + y_4)
$$
and
\[ ax_1 + 2ax_2 = a(x_1 + 2x_2) = a(x_3 + 2x_4) = ax_3 + 2ax_4. \]

Thus \( x + y \in U \) and \( ax \in U \). Therefore by the subspace criterion it follows that \( U \) is a subspace.

(b) Since \( 1 + 2 \cdot 0 = 1 = 1 + 2 \cdot 0 \) it follows that \( v_1 \in U \). Similarly, since \( 0 + 2 \cdot 1 = 2 = 0 + 2 \cdot 1 \) it follows that \( v_2 \in U \).

(c) Let \( a_1, a_2 \in \mathbb{R} \) such that
\[ a_1v_1 + a_2v_2 = 0. \]

Then it follows that \( a_1 = a_2 = 0 \) from the special form of the vectors \( v_1 \) and \( v_2 \). Thus \( N = \{v_1, v_2\} \) is a linear independent subset of \( U \).

(d) Let us first determine a basis of \( U \) and with this also the dimension of \( U \). We know that \( x \in U \) if and only if
\[ x_1 + 2x_2 - x_3 - 2x_4 = 0 \]
and this is the case if and only if
\[ x_1 = -2r + s + 2t \]
\[ x_2 = r \]
\[ x_3 = s \]
\[ x_4 = t \]
for some unique numbers \( r, s, t \in \mathbb{R} \). But this is again true if and only if
\[ x = r \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
for some unique numbers \( r, s, t \in \mathbb{R} \). Thus
\[ B := \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]
is a basis of \( U \). Since it contains 3 elements it follows that \( \dim(U) = 3 \). Thus any linear independent subset of \( U \) with 3 elements is a basis of \( U \). Therefore it is enough to look for a vector \( v_3 \), such that
\[ \{v_1, v_2, v_3\} \]
is a linear independent subset of \( U \) with precisely 3 vectors. Using the above results we may try with any of the vector of \( B \). For example set
\[ v_3 := \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

Now the set \( \{v_1, v_2, v_3\} \) has three elements. We have to verify that with this choice the set \( \{v_1, v_2, v_3\} \) is linear independent. Therefore consider the homogeneous linear equation
\[ a_1v_1 + a_2v_2 + a_3v_3 = 0 \quad (\ast) \]
in the unknown variables $a_1, a_2$ and $a_3$. Its simple coefficient matrix is

$$
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
$$

and we can transform this matrix using elementary row transformations as follows:

$$
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & -2 \\
0 & 0 & 1
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 0 & -2 \\
0 & 0 & 0
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

Thus from the last (or already from the 2-nd last) matrix we can conclude that there exists only the trivial solution to the linear equation $(\ast)$ and thus the set $\{v_1, v_2, v_3\}$ is indeed linear independent and therefore a basis of $U$ which contains by construction the set $N$.

(3) (a) Since $\text{rank}(v_1, \ldots, v_n) = n - 1$ it follows that $\{v_1, \ldots, v_n\}$ is a linear dependent set but there exists a linear independent subset $M$ of $\{v_1, \ldots, v_n\}$ with $n - 1$ elements. We may assume without any loss of generality that

$$M = \{v_1, \ldots, v_{n-1}\}.$$

Then $M$ is a linear independent generating system of $U := \text{span} M$ and therefore $M$ is a basis of $U$. Furthermore $v_n \in U$.

Assume that

$$a_1v_1 + \ldots + a_n v_n = 0 \quad (\ast)$$

is a non-trivial linear combination of the zero vector and assume hat

$$b_1v_1 + \ldots + b_{n-1} v_{n-1} = 0 \quad (\ast\ast)$$

is another linear combination of the zero vector, either trivial or not.

First we want to show that $a_n \neq 0$. To verify this claim we assume towards a contradiction that $a_n = 0$. Then $(\ast)$ reads

$$a_1v_1 + \ldots + a_{n-1} v_{n-1} = 0$$

and since this is a linear combination of the zero vector by linearly independent vectors it follows that $a_1 = \ldots = a_{n-1} = 0$. But this then implies then that $(\ast)$ is actually the trivial linear combination of the zero vector in contradiction with the assumption. Thus the assumption $a_n = 0$ is false and it follows that $a_n \neq 0$.

Thus $b_n/a_n$ is defined and we set $c := b_n/a_n$. Then $b_n = ca_n$. Now from $(\ast)$ and $(\ast\ast)$ follows that

$$ca_1v_1 + \ldots + ca_{n-1} v_{n-1} = -ca_n v_n = -b_n v_n$$

and

$$b_1v_1 + \ldots + b_{n-1} v_{n-1} = -b_n v_n$$

Since $M = \{v_1, \ldots, v_{n-1}\}$ is a basis of $U$ it follows that then necessarily also $b_i = ca_i$ for $1 \leq i \leq n - 1$ and therefore altogether $b_i = ca_i$ for $1 \leq i \leq n$.

(b) The solution space $M_0$ is one dimensional.
(4) Assume towards a contradiction that \( C^0(\mathbb{R}) \) is finitely generated. Then \( C^0(\mathbb{R}) \) must have a finite basis and thus \( C^0(\mathbb{R}) \) must be \( n \)-dimensional for some \( n \in \mathbb{N} \). Consider the subset
\[
\{1, x, x^2, \ldots, x^n\} \subset C^0(\mathbb{R})
\]
which consists of \( n + 1 \) elements. We know from exercise (4) from the 3-rd exercise sheet that this set is linearly independent. But this is a contradiction because a linear independent subset of a \( n \)-dimensional space has at most \( n \) elements. Thus the assumption that \( C^0(\mathbb{R}) \) is finitely generated is proven to be wrong and it follows that \( C^0(\mathbb{R}) \) is not finitely generated.

(5*) (a) Let \( a = (a_i)_{i \in \mathbb{N}} \) and \( b = (b_i)_{i \in \mathbb{N}} \) be two Fibonacci sequences, that is \( a, b \in F \) and let \( c \in \mathbb{R} \) be a real number. We have to show that \( a + b \in F \) and \( ca \in F \). Let \( i \in \mathbb{N} \). Then
\[
(a + b)_{i+2} = a_{i+2} + b_{i+2}
= a_{i+1} + a_i + b_{i+1} + b_i
= (a + b)_{i+1} + (a + b)_i
\]
and
\[
(ca)_{i+2} = ca_{i+2}
= c(a_{i+1} + a_i)
= (ca)_{i+1} + (ca)_i.
\]
Therefore \( a + b \) and \( ca \) are again Fibonacci sequences. Moreover the constant zero sequence is a Fibonacci sequence and this \( F \) is not empty. Therefore we can apply the subspace criterion and \( F \) is a linear subspace of \( \mathbb{R}^N \).

(b) Observe that a Fibonacci sequence is entirely and uniquely determined by the first two elements in the sequence. Thus any two sequences \( a, b \in F \) such that
\[
\{ (a_0), (b_0) \}
\]
is a basis of \( \mathbb{R}^2 \) will do as a basis of \( F \). Thus \( \dim F = 2 \).
Note that this result means that \( F \cong \mathbb{R}^2 \).

(c) We have to find all \( x \in \mathbb{R} \) such that \( a_i := x^i \) defines a Fibonacci sequence. That is we look for all \( x \in \mathbb{R} \) such that
\[
x_{i+2} = x_{i+1} + x_i
\]
for all \( i \in \mathbb{N} \). Since \( x^0 \) is not defined for \( x = 0 \) we have as the first requirement that \( x \neq 0 \). In this case we can divide the equation (\( * \)) by \( x^i \) and get
\[
x^2 = x + 1.
\]
This quadratic equation has two solutions, namely
\[
x_1 := \frac{1 + \sqrt{5}}{2} \approx 1.618033988749894848
\]
and
\[
x_2 := \frac{1 - \sqrt{5}}{2} \approx -0.618033988749894848.
\]
Thus there exists two sequences which are of the form \( a_i := x^i \), namely
\[
\alpha := (x_1^i)_{i \in \mathbb{N}} \quad \text{and} \quad \beta := (x_2^i)_{i \in \mathbb{N}}.
\]
We need to show that those sequences are linearly independent. Assume that \(a, b \in \mathbb{R}\) such that
\[
a \alpha + b \beta = 0.
\]
Then
\[
0 = a \alpha_0 + b \beta_0 \\
= a x_1^0 + b x_2^0 \\
= a + b
\]
and
\[
0 = a \alpha_1 + b \beta_1 \\
= a x_1 + b x_2 \\
= a x_1 - a x_2 \quad \text{(due to (**))} \\
= a (x_1 - x_2) \\
= a \left(1 + \sqrt{5} \over 2 - 1 - \sqrt{5} \over 2\right) \\
= a \sqrt{5}.
\]
From (***), follows that \(a = 0\) and then from (**) that also \(b = 0\). Thus \(\alpha\) and \(\beta\) are linearly independent elements of \(F\) and since \(F\) is a two dimensional vector space, it follows that \(\{\alpha, \beta\}\) forms a basis of \(F\).

(d) Since the sequences \(\alpha\) and \(\beta\) of the previous part forms a basis of \(F\), we can write every element \(v \in F\) as a linear combination of \(\alpha\) and \(\beta\), that is
\[
v = t_1 \alpha + t_2 \beta
\]
for some unique real numbers \(t_1\) and \(t_2\) which depend only on \(v\) and \(v\) is uniquely determined by its first two values \(v_0\) and \(v_1\). Then this equation yields the following system of two linear equations
\[
t_1 + t_2 = v_0 \\
x_1 t_1 + x_2 t_2 = v_1
\]
in the unknown variables \(t_1\) and \(t_2\) (which is derived from (†) using the first two elements of the sequences on both sides). We know already that it has a unique solution (since \(\{\alpha, \beta\}\) is a basis of \(F\)), we just need to determine it. We can – for example – use the third case of Proposition 1.1:
\[
t_1 = \frac{\delta(v_0, 1, v_1, x_2)}{\delta(1, 1, x_1, x_2)} = \frac{v_0 x_2 - v_1}{x_2 - x_1} \\
t_1 = \frac{\delta(1, v_0, x_1, v_1)}{\delta(1, 1, x_1, x_2)} = \frac{v_1 - v_0 x_1}{x_2 - x_1}
\]
Thus we get – due to \(v_i = t_1 \alpha_i + t_2 \beta_i\) – for the \(i\)-th value of the arbitrary Fibonacci sequence \(v = (v_i)_{i \in \mathbb{N}}\), the following expression
\[
v_i = \frac{v_0 x_2 - v_1}{x_2 - x_1} x_i^1 + \frac{v_1 - v_0 x_1}{x_2 - x_1} x_i^2
\]
in terms of the first two numbers \(v_0\) and \(v_1\) only.

Note how we used the methods of Linear Algebra to deduce this compact formula for any Fibonacci sequence!