(1) **Linear Maps.** In this task \( V \) and \( W \) are vector spaces over a field \( F \).

(a) What is by definition a linear map \( f : V \to W \)?
(b) What is \( \text{im} f \) and \( \ker f \)?
(c) Prove that \( \ker f \) is a subspace of \( V \).
(d) What can you conclude from \( \ker f = 0 \)?
(e) Assume that \( V \) is a finite dimensional vector space. What does the dimension formula for linear maps state?
(f) Assume that \( V \) and \( W \) are finite dimensional vector of the same dimension and let \( f : V \to W \) be a linear map. Prove that the following three statements are equivalent:
   (i) \( f \) is an isomorphism.
   (ii) \( f \) is a monomorphism.
   (iii) \( f \) is an epimorphism.

(2) **Coordinate Description of Linear Maps.** In this task \( V \) and \( W \) are finite dimensional \( F \)-vector spaces. Assume that \( B = (b_1, \ldots, b_n) \) is a basis of \( V \) and that \( C = (c_1, \ldots, c_m) \) is a basis of \( W \).

(a) Let \( f : V \to W \) be a linear map. How are the coefficients of the coordinate matrix \( A \) of \( f \) with respect to the bases \( B \) and \( C \) defined?
(b) What is a transition matrix and what they used used for? Give a short explanation.
(c) What is the definition of two \( m \times n \)-matrices \( A \) and \( A' \) being equivalent?
(d) If \( A \) and \( A' \) are equivalent matrices, what can you conclude from it? What can you conclude in the special case that \( A \) is an \( n \times n \)-matrix which is equivalent to the identity matrix \( I \)?
(e) Consider the endomorphism \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[
A := \begin{pmatrix} -8 & -2 & -7 \\ -15 & -3 & -17 \\ 0 & 0 & 1 \end{pmatrix}
\]

What is the coordinate matrix \( A' \) of \( A \) with respect to the basis

\[
B := \begin{pmatrix} 2 \quad 3 \quad 1 \\ 5 \quad 7 \quad 2 \\ 0 \quad 1 \quad 0 \end{pmatrix}
\]

of \( \mathbb{R}^3 \)?
(f) What all can you conclude about the endomorphism \( A \) from the coordinate matrix \( A' \) obtained in part (e)?
(3) **Transition Matrices** In this task we consider the $F^n$ vector space $F^n$. Assume that $B$ and $C$ are arbitrary bases of $F^n$.

(a) Let $S$ be the transition matrix from canonical standard basis of $F^n$ to $B$ and let $T$ be the transition matrix from the canonical standard basis of $F^n$ to $C$.

Derive a formula for the transition matrix from the basis $C$ to the basis $B$ using the matrices $S$ and $T$.

(b) Consider the bases

$$B := \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

and

$$C := \left( \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

of the vector space $\mathbb{R}^3$.

Compute the transition matrix from the basis $C$ to the basis $B$.

(4) **Rank of a Linear Map.** In this task $V$ and $W$ are vector spaces over the same field $F$ and $f: V \rightarrow W$ a linear map.

(a) How is the rank of the linear map $f$ defined?

(b) Assume that $u_1, \ldots, u_k$ is a system of vector space. What relation exists between $\text{rank}(u_1, \ldots, u_k)$ and $\text{dim}\text{span}\{u_1, \ldots, u_k\}$?

(c) Assume from now on that $V$ and $W$ are finite dimensional vector spaces. Let $B := (b_1, \ldots, b_m)$ be a basis of $V$ and $C := (c_1, \ldots, c_m)$ be a basis of $W$. What is the basis isomorphism of $W$ with respect to the basis $C$? How is the coordinate isomorphism of $W$ with respect to the basis $C$ defined?

(d) Assume that $u_1, \ldots, u_k$ is a system of vectors of the vector space $W$. Let $a_1, \ldots, a_k \in F^m$ be the coordinate vectors of the vectors $u_1, \ldots, u_k$ with respect to the basis $C$. Why does the equality $\text{rank}(u_1, \ldots, u_k) = \text{rank}(a_1, \ldots, a_k)$ hold?

(e) Assume that $A$ is the coordinate matrix of $f$ with respect to the bases $B$ and $C$. Show that $\text{rank} f = \text{rank} A$. 

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